THE EDDINGTON APPROXIMATION

AS 712 Project
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I. INTRODUCTION

The Eddington Approximation, first proposed by Sir Arthur Stanley Eddington, is one of the approximations astronomers made when trying to simply the process of solving the radiative transfer equation (given below as eqn 1) to understand some physical conditions inside stellar atmospheres, such as temperature, density, pressure and intensity.

$$\frac{dI_\lambda}{d\tau_\lambda} = I_\lambda - S_\lambda \tag{1}$$

The approximations greatly simplify the derivation for any quantity but still gives an accurate picture of what is happening inside the stellar atmospheres. In this report, we are interested in the concept of optical depth of $\frac{2}{3}$ in stellar atmospheres and how using the Eddington Approximation gives us this result and briefly discuss its consequences.

II. ASSUMPTIONS

1. The first of the four main assumptions being made is that the stellar atmosphere is in Local Thermal Equilibrium (LTE). Under this assumption, we can take the source function to be equal to the Planck function of a blackbody, $S_\lambda = B_\lambda$.
2. The second assumption here is that we assume the atmosphere here we are dealing with is a Plane-Parallel Atmosphere. By that, we mean the radius of curvature of the atmosphere is much greater than its thickness so essentially we can treat the top of the atmosphere to be parallel to the bottom of the atmosphere. To obtain a meaningful measure of distance inside the atmosphere, we introduce vertical optical depth τ_λ ,which is the optical depth measured perpendicular to the plane of atmosphere from the bottom to the top. The relation between vertical optical depth and optical depth is given by $\tau_\lambda = \tau_{\lambda,v} \sec\theta$, so the radiative transfer equation will now transform to

the following form:

$$\cos\theta \frac{dI_\lambda}{d\tau_{\lambda,\upsilon}} = I_\lambda - S_\lambda \tag{2}$$

3. The third assumption here is that we are assuming the atmosphere is a gray atmosphere, meaning the opacity in the atmosphere is independent of the wavelength. Now, we may remove the wavelength dependencies by writing τ_ν instead of $\tau_{\lambda,\nu}$ and integrating eqn (2) over all wavelength to drop the wavelength subscript. Now we got the transfer equation without any wavelength dependencies.

4. One final and crucial assumption is the Eddington Approximation. It is worth to be noted here that Eddington Approximation is a special case of two-stream approximation where we assume the intensity of the radiation goes in two directions, one as I_{out} which is in the outward z direction and the other as I_{in} which is in the inward -z direction. In addition, $I_{in} = 0$ at the top of the atmosphere where the vertical optical depth is zero. From this, we have the total intensity as I = I_{out} + I_{in}. Taking the average of the total intensity, we have

$$< I >= \frac{1}{2}(I_{out} + I_{in}) \tag{3}$$

III. DERIVATION

To start off, we integrate eqn (2) without the wavelength dependencies over all solid angles and we have

$$\frac{d}{d\tau_\nu}\int I\cos\theta d\Omega = \int I d\Omega - S\int d\Omega \tag{4}$$

The source function comes out of the integral because it does not depend on any directions. Using the following two equations, we find eqn (5).

$$< I_\lambda >= \frac{1}{4\pi}\int I_\lambda d\Omega = \frac{1}{4\pi}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} I_\lambda \sin\theta d\theta d\phi$$

$$F_\lambda d\lambda = \int I_\lambda d\lambda \cos\theta d\Omega = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} I_\lambda d\lambda \cos\theta \sin\theta d\theta d\phi$$

$$\frac{dF_{rad}}{d\tau_\upsilon} = 4\pi(< I > -S) \tag{5}$$

Now going back to eqn(2) again without wavelength dependencies, this time we multiply both sides by $\cos\theta$ and integrate over all solid angles. We have

$$\frac{d}{d\tau_\nu}\int I\cos^2\theta d\Omega = \int I\cos\theta d\Omega - S\int\cos\theta d\Omega \tag{6}$$

Recall that radiation pressure is defined as $P_{rad}d\lambda = \frac{1}{c}\int I\cos^2\theta d\Omega$ so the left hand side of eqn (6) is simply just radiation pressure times the speed of light. The second integral goes to zero when evaluated and according to eqn(5), the first integral gives radiative flux. So putting together eqn (5) and (6), we have $\frac{dP_{rad}}{d\tau_\upsilon} = \frac{1}{c}F_r$. Integrating this equation we now have

$$P_{rad} = \frac{1}{c}F_{rad}\tau_\upsilon + C \tag{7}$$

where C is just a constant coming out of the integral, which we will solve later.

Now using the crucial Eddington Approximation to finish the rest of the derivation. In this approximation, the mean intensity is given by eqn(3), radiative flux and radiation pressure are give by the following

$$F_{rad} = \pi(I_{out} - I_{in}) \tag{8}$$

$$P_{rad} = \frac{2\pi}{3c}(I_{out} + I_{in}) = \frac{4\pi}{3c} < I > \tag{9}$$

Inserting eqn(9) into eqn(7) we have $\frac{4\pi}{3c} < I >= \frac{1}{c}F_{rad}\tau_\upsilon + C$. To get the constant, we may evaluate eqn(3) and eqn(8) at the top of the atmosphere where I_{in} is zero. Plugging into the previous unlabeled equation we have the constant $C = \frac{2}{3c}F_{rad}$ and eqn(7) now looks like

$$\frac{4\pi}{3} < I >= F_{rad}(\tau_\upsilon + \frac{2}{3}) \tag{10}$$

We already know radiative flux is expressed by $F_{rad} = \sigma T_e^4$ where T_e is the effective temperature and since we are in LTE, there is no net energy transfer so this flux is a constant at any level inside the atmosphere. This implies that the mean intensity is also equal to the source function, and we have the following relation

$$< I >= S = B \tag{11}$$

Because $F = \pi B = \pi < I >$, plugging this into eqn(10) and we have the final expression which gives the temperature profile inside the stellar atmosphere and this completes our derivation.

$$T^4 = \frac{3}{4} T_e^4 (\tau_v + \frac{2}{3}) \tag{12}$$

IV. CONSEQUENCES

Looking at eqn(12), if we try to balance the actual temperature with the effective temperature, we would find out that this occurs when $\tau_v = \frac{2}{3}$. By definition, the surface of a star has a temperature equal to the effective temperature. This equation implies that the surface of the star does not actually lie on top of the stellar atmosphere. Instead, the surface lies deeper down at a vertical optical depth of two-thirds. This means, the photons that are coming towards us were originated approximately at this depth averaged over the whole disk inside the stellar atmosphere and thus this is how astronomers define where the photosphere is.

One phenomenon that we see because of this is limb darkening of the Sun. When we look closer at the Sun, its disk appears to be brighter at the center than the edge or the limb. It can be understood as when we look at the edge of the disk, we cannot see to the same optical depth as when we look at the center. We are essentially looking into less material when we look at the limb. Because of the temperature profile in eqn (12), the temperature in a typical stellar atmosphere decreases when the height or radius increases. This means the outer part of the atmosphere has a lower temperature and are thus radiating at a lesser level of intensity. That explains why the limb of the Sun appears to be darker.

REFERENCES

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