LIMB DARKENING

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ABSTRACT. Limb darkening is the effect seen in stars where the edge, or limb, of the star appears to be dimmer than the rest of the star. The effect of limb darkening can be seen in optical images of the Sun. Using the equation of radiative transfer, we explore the mathematical description of limb darkening by deriving an expression for the emergent intensity at the surface of a star. A more intuitive explanation is also given by considering a temperature gradient in the layers of a star.

1. Emergent Intensity

It is possible to obtain an expression for the emergent intensity from the surface of a star as a function of a viewing angle, measured from the center of the star. We start with the equation of radiative transfer.

$$\frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + j_\nu$$

Dividing by α_ν gives us an equation that depends on the optical depth τ_ν, rather than the distance s.

$$\frac{dI_\nu}{d\tau_\nu} = I_\nu - S_\nu$$

Where S_ν is the source function. We multiply through by $e^{-\tau_\nu}$ to obtain an expression that can be integrated.

$$\frac{dI_\nu}{d\tau_\nu} e^{-\tau_\nu} = I_\nu e^{-\tau_\nu} - S_\nu e^{-\tau_\nu}$$

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$$\frac{d(I_\nu e^{-\tau_\nu})}{d\tau_\nu} = -S_\nu e^{-\tau_\nu}$$

$$\int_{\tau_{\nu,0}}^{0} d(I_\nu e^{-\tau_\nu}) = -\int_{\tau_{\nu,0}}^{0} S_\nu e^{-\tau_\nu} d\tau_\nu$$

After integration and solving for the intensity at the surface of the star, where $\tau_\nu = 0$, we arrive at an expression for the emergent intensity.

$$I_\nu(0) = I_{\nu,0} e^{-\tau_{\nu,0}} - \int_{\tau_{\nu,0}}^{0} S_\nu e^{-\tau_\nu} d\tau_\nu$$

Now we assume a plane-parallel atmosphere.

$$I_\nu(0) = I_{\nu,0} e^{-\tau_{\nu,0}\sec\theta} - \int_{\tau_{\nu,0}\sec\theta}^{0} S_\nu \sec\theta e^{-\tau_\nu \sec\theta} d\tau_\nu$$

To get contribution from all layers of the star, we let $\tau_\nu \to \infty$.

$$I_\nu(0) = \int_{0}^{\tau_{\nu,0}\sec\theta} S_\nu \sec\theta e^{-\tau_\nu \sec\theta} d\tau_\nu$$

In order to arrive at a final expression, we must make an assumption about the form of our source function S_ν. One "guess" at the form of the source function is one that is linear with respect to the optical depth. Our source function becomes $S_\nu = a_\nu + b_\nu \tau_\nu$. This form of S_ν makes sense, as one would expect the amount of radiation leaving the interior of the star to increase as a function of the optical depth.

$$I_\nu(0) = \int_{0}^{\tau_{\nu,0}\sec\theta} (a_\nu + b_\nu \tau_\nu) \sec\theta e^{-\tau_\nu \sec\theta} d\tau_\nu$$

After integration, we arrive at our final expression for the emergent intensity.

$$I_{\nu,0}(\theta) = a_\nu + b_\nu \cos\theta$$

This expression describes what we see in the optical images of our Sun; a maximum intensity when looking at the center of the Sun ($\theta = 0$), which then decreases as we look away from the center ($\theta \geq 0$).

2. Limb Darkening Explained Using The Eddington Approximation

The Eddington Approximation gives us the following expression for the source function, which is the same as the mean intensity.

$$S = \langle I \rangle = \frac{\sigma T^4}{\pi} = \frac{3\sigma}{4\pi} T_e^4 \left(\tau_\nu + \frac{2}{3} \right)$$

We can get this into the form of the source function $S_\nu = a_\nu + b_\nu \tau_\nu$ by letting

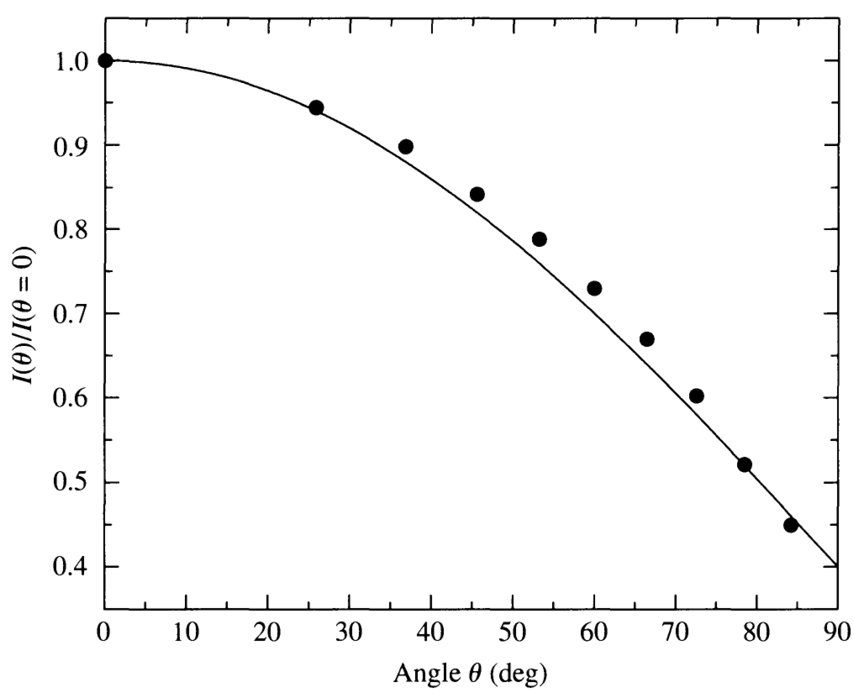
$$a_\nu = \frac{\sigma}{2\pi} T_e^4 \quad \text{and} \quad b_\nu = \frac{3\sigma}{4\pi} T_e^4.$$

The expression for the emergent intensity becomes

$$I_0(\theta) = \frac{\sigma}{2\pi} T_e^4 + \frac{3\sigma}{4\pi} T_e^4 \cos\theta$$

We can see how the emergent intensity behaves as a function of the viewing angle by taking $\frac{I_0(\theta)}{I_0(\theta=0)}$.

$$\frac{I_0(\theta)}{I_0(\theta = 0)} = \frac{2}{5} + \frac{3}{5} \cos\theta$$



The solid line in the figure is the previous relation of the emergent intensity at an angle θ. Superimposed is actual data taken from the face of our Sun. Though it is not a perfect fit, our initial guess at the form of the source function is an acceptable model for the effect of limb darkening.

3. A More intuitive Apporach

From the previous equation, derived from the Eddington Approximation, it is clear that what we call the surface of a star occurs at an optical depth of $\frac{2}{3}$. This means that the light that we receive is coming from a layer inside the star where $\tau = \frac{2}{3}$. Because there is a temperature gradient inside of a star, we will receive different amounts of light when viewing a stars center and viewing the edge of a star. This is illustrated clealy in the following figure.