# The Eddington-Barbier Approximation

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AS 712 Project Report

December 13, 2016

**Abstract**

The Eddington-Barbier Approximation is a simple radiative transfer approximation that assumes atmosphere qualities and then calculates the resulting observable quantities. This short report goes through the simple derivation of the formulas and gives a brief discussion of the resulting consequences. The main consequence is the common understanding the we see to an average optical depth of $\tau = \frac{2}{3}$ and the phenomenon of limb darkening.

## 1 What is the Eddington-Barbier Approximation?

The Eddington-Barbier Approximation (EB approximation) is a simple approximation for radiative transfer in a simplified stellar atmosphere given by Sir Arthur Eddington, an English astronomer, and Joseph Émile Barbier, a French astronomer. This approximation uses a few simple assumptions about the nature of a stellar atmosphere and then derives quantities for Intensity, Flux, Radiation Pressure, and a temperature profile.

The resulting equations give simple means of calculating important values and allows for a deeper understanding of the properties of stellar atmospheres. It also allows for an explanation of the obsessionally seen limb darkening effect.

## 2 Basic Approximations

In order to calculate some important quantities we need to make a few assumptions, which are listed here.

1. Plain Parallel Atmosphere Approximation
2. Grey Atmosphere
3. Local Thermodynamic Equilibrium (LTE)
4. Two-Stream Approximation

The first assumption is the plain parallel approximation. In this approximation we assume that the thickness of the atmosphere is much smaller then the radius of curvature of the object which is usually applicable when discussing stars. When using this approximation we can say that the radiative transfer equation will take the following form with $\tau_v$ being the vertical optical depth. This vertical optical depth is related to the true optical depth by the following relation: $\tau_v = \tau \cos\theta$ where $\theta$ is the angle from the ray to the surface normal.

$$\cos\theta \frac{dI_\nu}{d\tau_\nu} = I_\nu - S_\nu \tag{1}$$

The second assumption we consider is that the atmosphere is a grey atmosphere. For us, this means that the absorption $\kappa_\nu$ is not dependent on wavelength at least over the range on which the temperature changes in the atmosphere. This gives that $\kappa_\nu = \kappa$ from which we find that the optical depth is not dependent on wavelength $\tau_\nu = \tau$.

The third approximation is that the atmosphere is in LTE. The temperature does not change over small regions and can be considered in equilibrium. This gives us that the source function $S_\nu$ should be given by a blackbody of function $B_\nu(T)$.

Lastly, the fourth approximation is using the two-stream approximation. With this, we assume that the radiation is only moving in two directions: in and out. This gives us that the total intensity is

$I = I_{in} + I_{out}$. We take that $I_{in} = 0$ at an optical depth $\tau = 0$. We also assume that the intensity is isotropic.

With these four simple assumptions (which are pretty close to the true behavior in a star) we can derive the Eddington-Barbier approximation and find a temperature profile for the atmosphere.

# 3 Derivations

To begin the derivations we find that the intensity is given (using a two-stream approximation) by

$$I = I_{in} + I_{out} \tag{2}$$

From here we can calulate the average intensity by integrating over all solid angles.

$$< I >= \frac{1}{4\pi}\int I d\Omega = \frac{1}{4\pi}\int_0^{2\pi} d\phi \int_0^{\pi} (I_{in} + I_{out})\sin\theta d\theta = \frac{1}{2}\left[I_{in}\int_{\pi/2}^{\pi}\sin\theta d\theta + I_{out}\int_0^{\pi/2}\sin\theta d\theta\right]$$

This gives us the result

$$< I >= \frac{1}{2}(I_{in} + I_{out}) \tag{3}$$

We can now take the first moment of the intensity to get the flux

$$F = \int I\cos\theta d\Omega = \int_0^{2\pi} d\phi \int_0^{\pi}(I_{in} + I_{out})\sin\theta\cos\theta d\theta =$$
$$\int_0^{2\pi} d\phi \left[I_{in}\int_{\pi/2}^{\pi}\sin\theta\cos\theta d\theta + I_{out}\int_0^{\pi/2}\sin\theta\cos\theta d\theta\right]$$

This results in the flux being given by

$$F = \pi(I_{out} - I_{in}) \tag{4}$$

Taking the second moment of the intensity will lead us to a formula for the radiation pressure.

$$P_{\text{rad}} = \frac{1}{c}\int I\cos^2\theta = \frac{1}{c}\int_0^{2\pi} d\phi\left[I_{in}\int_{\pi/2}^{\pi}\sin\theta\cos^2\theta d\theta + I_{out}\int_0^{\pi/2}\sin\theta\cos^2\theta d\theta\right]$$

This results in the following two expressions (using Equation (3)) for the radiation pressure

$$P_{\text{rad}} = \frac{2\pi}{3c}[I_{in} + I_{out}] = \frac{4\pi}{3c} < I > \tag{5}$$

We can find another expression for the radiation pressure by using the radiative transfer equation. Since we are assuming a grey atmosphere, we can remove the frequency dependence from the radiative transfer equation.

$$\cos\theta\frac{dI}{d\tau} = I - S \tag{6}$$

We can then integrate over all solid angle by the following, while pulling the $\cos\theta$ in the integral.

$$\frac{d}{d\tau}\int I\cos\theta d\Omega = \int I d\Omega - S\int d\Omega \tag{7}$$

However, the left hand side of this equation contains the first moment of the intensity, so we can rewrite this as

$$\frac{dP_{\text{rad}}}{d\tau} = 4\pi[< I > -S] \tag{8}$$

From this, we can integrate to find another expression for radiation pressure as follows,

$$P_{\text{rad}} = \frac{1}{c}F\tau + C \tag{9}$$

where $C$ is a constant of integration. We can use our boundary condition to find this constant. Since $I_{in} = 0$ at $\tau = 0$ we can say that

$$< I_{out}(\tau = 0) >= \frac{I_{out}}{2}$$

Then using out expression for flux (Equation (4)) we say that

$$< I_{out}(\tau = 0) >= \frac{F}{2\pi}$$

Setting Equation (5) at $\tau = 0$ we get that

$$\frac{4\pi}{3c} < I >= \frac{4\pi}{3c}\frac{F}{2\pi} = \frac{1}{c}F + C$$

Solving for the integration constant gives us,

$$C = \frac{2F}{3c} \tag{10}$$

We can now write the full expression for the radiation pressure

$$P_{\text{rad}} = \frac{1}{c}F\tau + \frac{2F}{3c} = \frac{F}{c}\left[\tau + \frac{2}{3}\right] = \frac{4\pi}{3c} < I > \tag{11}$$

Rearranging for the average intensity we get

$$< I >= \frac{3F}{4\pi}\left[\tau + \frac{2}{3}\right] \tag{12}$$

We can then use the Stephan-Boltzmann law to relate flux to effective temperature.

$$< I >= \frac{3\sigma T^4_{\text{eff.}}}{4\pi}\left[\tau + \frac{2}{3}\right] \tag{13}$$

However, from assumption number 3, assuming LTE, we know that the average intensity we observe should be blackbody, given by

$$< I >= S = B = \frac{\sigma T^4}{\pi} \tag{14}$$

We finally get a result for the temperature.

$$T^4 = \frac{3}{4}\left[\tau + \frac{2}{3}\right] T^4_{\text{eff.}} \tag{15}$$

Equation (13) and Equation (15) are the main resulting equations from the Eddington-Barbier approximation. They describe the average intensity and temperature we observe for a specific optical depth.

# 4 Consequences

The main formula of interest that comes out of our derivation of the EB approximation is the temperature profile that we found.

$$T^4 = \frac{3}{4}\left[\tau + \frac{2}{3}\right] T^4_{\text{eff.}} \tag{16}$$

This profile shows the dependence of the temperature on the vertical optical depth. Given a optical depth, we can (approximately) find the corresponding temperature of the region at that optical depth. However the most important consequence comes from the fact there is a temperature gradient which has the temperature decreasing outward through the atmosphere.

Lets consider our star to be made of concentric shells (regions) in which the temperature decreases outward according to our temperature profile. If we examine the intensity of light coming from increasing distances to the center of the star we can observe the main consequence of the EB approximation: limb darkening.

Limb darkening is the effect that the outer regions of a star (its limb) is less luminous then the inner (center) part of the star. As we look at the outer edges of a star, we are looking at a high angle to the stars surface. This leads us to look at material that is farther from the center of the star than if we were looking at the center of the star. Since this material is closer to the surface of the star, we can see from our temperature profile that this material is also has a cooler temperature. We know (since we are considering our star to be in LTE so it radiates as a blackbody) that material at a cooler temperature also has a lower luminosity. Since this cooler, less luminous material is being seen next to hotter, more luminous material near the center of the star, contrasting makes it appear darker.

We can take this one step further by using our temperature profile and the Stephan-Boltzmann law. From the Stephan-Boltzmann law we know that the flux we observe is coming from material at a corresponding effective temperature $T_{\text{eff.}}$. The effective temperature is not the actual temperature of the star but just of the material we are seeing. From the temperature profile, we see that there is a direct relation between effective temperature and the actual temperature. It comes directly from the temperature profile that the effective temperature equals the actual temperature of the material when the optical depth of the material is $\tau = \frac{2}{3}$.

This is a remarkable result. We now see that the flux we are observing is not coming from the traditional surface of the star where $\tau = 0$, but rather from further down at an optical depth of $\tau = \frac{2}{3}$ which is now how we define the photosphere or "surface" of a star.

We can also see that if we are looking at an optical depth of $\tau = \frac{2}{3}$ then then average intensity that we observe is

$$< I >= \frac{\sigma T_{\text{eff.}}^4}{\pi} = B \tag{17}$$

The average intensity is the blackbody function, which makes sense if we are considering each layer to be in LTE.