AS712 Final Project: **Bremsstrahlung**

Adham El-Batal

December 8, 2016

Abstract

Bremsstrahlung is the physical process by which electrons and ions undergo free-free interactions resulting in a deceleration and subsequent emission of radiation. It is a prominent probe of several astrophysics processes, both relativistic and non-relativistic. We observe bremsstrahlung radiation in the form of radio waves in HII regions as well as X-ray emission from low mass X-ray binaries (LMXB) and the intracluster medium (ISM). In this paper, I outline the underlying physics including relativistic corrections and present the uses and relevance of bremsstrahlung in astronomy.

Introduction & basic underlying physics

The term *bremsstrahlung* is German for "braking radiation." It is, quite literally, the radiation that arises from a charged particle "braking" or decelerating in the vicinity of an oppositely charged particle.

I will consider it from the perspective of an influx of rapidly moving electrons past stationary ions, giving rise to a spectrum of radiation. While bremsstrahlung from decelerating ions is theoretically possible, due to considerations that will arise later, I will ignore it for this text.

To proceed with the rest of this text, I will first venture to *briefly* describe and derive the processes by which charged particles radiate. For the sake of brevity, I provide these in a "back-of-the-envelope" context as not to detract from the main topic of bremsstrahlung radiation. I will then expand the effective results to a thermal case, where a Maxwellian distribution of velocities is assumed.

If we imagine charged particles approximated as a dipole, their dipole moment can be expressed as $\mathbf{d}(t) = \sum q_i \mathbf{r}_i(t)$. We can express this dipole moment in frequency space through a Fourier transform such that

$$\hat{\mathbf{d}}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathbf{d}(t) e^{i\omega t} \mathrm{dt}$$

which can be extended to an accelerating situation such that

$$\hat{\mathbf{d}}(\omega) = \frac{1}{2\pi\omega^2} \int_{-\infty}^{\infty} \ddot{\mathbf{d}}(t) e^{i\omega t} \mathrm{dt}$$

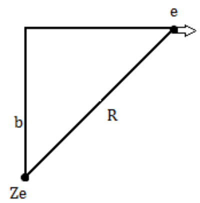
given that the dipole moment is a periodic oscillatory function (sinusoidal). This is given that $\ddot{\mathbf{d}} = \sum q \ddot{\mathbf{r}}_i(t) = \sum q \dot{\mathbf{v}}_i(t)$ Furthermore, recalling that

$$\frac{dW}{d\omega} = \frac{8\pi\omega^4}{3c^3} |\hat{\mathbf{d}}(\omega)|^2$$

we can derive a form for the radiation power for bremsstrahlung.

I will proceed to do this classically and implement the appropriate corrections for quantum properties. That will serve to adequately represent bremsstrahlung radiation.

Classical derivation of bremsstrahlung



Above we see a simple schematic of the simplest situation in which bremsstrahlung radiation arises. An electron moves with some velocity (represented in the figure above with an arrow) past an ion of charge Ze where Z denotes the atomic number. The particles are a distance R apart and are separated by some impact parameter b.

The dipole moment in this case would be $\mathbf{d} = e\mathbf{R}$, which then follows that $\ddot{\mathbf{d}} = -e\ddot{\mathbf{R}} = -e\dot{\mathbf{v}}$. The Fourier transform in the preceding section therefore becomes

$$\hat{\mathbf{d}}(\omega) = \frac{e}{2\pi\omega^2}\int_{-\infty}^{\infty} \dot{\mathbf{v}} e^{i\omega t} \mathrm{d}t$$

To approximate this, we can have the electron move very rapidly past the ion, such that any angular deviation is minimal. The interaction between the two particles would take over a time that we can call the collision time, simply given as the impact parameter divided by the velocity of the electron ($\tau = b/v$). This small angle approximation allows us to simply take the asymptotic limits of the dipole moment instead of rigorously evaluating the above integral. It follows that:

$$\lim_{\omega\tau\ll 1} \hat{\mathbf{d}}(\omega) = \frac{e}{2\pi\omega^2}\Delta\mathbf{v} \quad \text{and} \quad \lim_{\omega\tau\gg 1} \hat{\mathbf{d}}(\omega) = 0$$

Given that $\Delta\mathbf{v} = \int_{-\infty}^{\infty} \dot{\mathbf{v}} \mathrm{d}t$, and only considering the component of the acceleration due to the Coulombic force along the normal of the electron's motion, we can obtain the expression

$$\Delta v = \frac{Ze^2}{m}\int_{-\infty}^{\infty} \frac{b}{(b^2 + v^2t^2)^{3/2}} dt = \frac{2Ze^2}{mbv}$$

we can conclude the power radiated per unit frequency in the realm where $b \ll v/\omega$ is in fact

$$\frac{dW}{d\omega} = \frac{8Z^2e^6}{3\pi c^3 m^2 v^2 b^2}$$

and approximately zero in another realms.

This situation, however, is merely for an interaction between one ion and one electron. To expand this to a volume consisting of n_e electrons per unit volume and n_i ions per unit volume, we can analogously say that

$$\frac{dW}{d\omega dV dt} = n_e n_i 2\pi v \int_{b_{min}}^{b_{max}} \frac{dW}{d\omega} b \, \mathrm{d}b$$

for a volume where the differential volume element is a simply $2\pi b \, \mathrm{d}b$ for a flux of electron $n_e v$ on the stationary ions.

The limits of the integral can be defined such that:

- $b_{\max}$ is where the integral reaches its low asymptotic limit of zero. A very rough but ultimately useful limit would be

$$b_{\max} = v/\omega$$

- The lower limit is where the small angle approximation given above is no longer valid. That would happen at

$$b_{\min} = \frac{4Ze^2}{\pi m v^2}$$

There are further quantum considerations for the effective minimum limit but those will be added *ad hoc* later for simplicity and brevity.

Finally, we get the end result of this classical single velocity approximation of bremsstrahlung that yields a power radiated per unit frequency per unit volume per unit time that is

$$\frac{dW}{d\omega dV dt} = \frac{16\pi e^6}{3\sqrt{3}c^3 m^2 v} n_e n_i Z^2 g_{ff}$$

where g_{ff} is the so-called Gaunt factor. It is a correction factor that accounts for further quantum effects and depends on the energy and frequency of the situation. The main consideration to pull away from here is that the power radiated (denoted below as P) is that such

$$P \propto \frac{n_e n_i Z^2}{m^2 v}$$

From this we can also conclude that while bremsstrahlung from the deceleration of an ion is possible, it needs on the order of 10^6 more power since the power radiated goes as $1/m^2$ and a proton is 2000 times heavier than an electron. Therefore, we can neglect that.

Thermal correction

So far we had only considered the case where the electron flux is moving at the same uniform speed. In reality, the electrons will have a distribution of velocities. It is safe to assume that this distribution will follow a Maxwellian pattern, dependent on temperature.

Therefore, we can infer that the probability of finding a particle between the velocities v and $v + \mathrm{d}v$ is proportional to the functional form:

$$v^2 \exp\left(\frac{-mv^2}{k_B T}\right) \mathrm{d}v$$

As such, we can extend our consideration of bremsstrahlung to a thermal distribution of electrons such that: