PHY3221 Detweiler Lecture #4

Relativity

Lorentz transformation

A coordinate rotation:

$$\begin{aligned} t' &= t \\ x' &= x\cos\theta + y\sin\theta \\ y' &= -x\sin\theta + y\cos\theta \\ z' &= z' \end{aligned}$$

Notice that *distance* is *conserved* under a coordinate rotation rotation:

$$\begin{aligned} x'^2 + y'^2 &= (x\cos\theta + y\sin\theta)^2 + (-x\sin\theta + y\cos\theta)^2 \\ &= x^2\cos^2\theta + 2xy\cos\theta\ \sin\theta + y^2\sin^2\theta + x^2\sin^2\theta - 2xy\cos\theta\ \sin\theta + y\cos^2\theta \\ &= x^2(\cos^2\theta + \sin^2\theta) + y^2(\sin^2\theta + \cos\theta) \\ &= x^2 + y^2. \end{aligned}$$

This might seem obvious because the mathematics just confirms your everyday experiences.

The Galilean transformation:

$$\begin{aligned} t' &= t \\ x' &= x - vt \\ y' &= y \\ z' &= z \end{aligned}$$

Inverse transformation:

$$\begin{aligned} t &= t' \\ x &= x' + vt' \\ y &= y' \\ z &= z' \end{aligned}$$

The Lorentz transformation:

$$\begin{aligned} t' &= \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}} \\ x' &= \frac{x - vt}{\sqrt{1 - v^2/c^2}} \\ y' &= y \\ z' &= z \end{aligned}$$

Inverse transformation:

$$\begin{aligned} t &= \frac{t' + vx'/c^2}{\sqrt{1 - v^2/c^2}} \\ x &= \frac{x' + vt'}{\sqrt{1 - v^2/c^2}} \\ y &= y' \\ z &= z' \end{aligned}$$

Notice that in the limit that $v/c \to 0$, but v remains finite, the Lorentz transformations approach the Galilean transformation. So, only when v is comparable to c are the effects of special relativity revealed.

Derive time dilation from the Lorentz transformations:
Two events, #1 at (t_1, x_1) and #2 at (t_2, x_2), with occur at the same place $(x_1 = x_2)$ in the t, x coordinate system. Thus the *proper time* between the events is

$$\Delta T_0 = t_2 - t_1.$$

The time between the events in the primed coordinate system is

$$\begin{aligned} \Delta T' &= t_2' - t_1' \\ &= \frac{t_2 - vx_2/c^2}{\sqrt{1 - v^2/c^2}} - \frac{t_1 - vx_1/c^2}{\sqrt{1 - v^2/c^2}} \\ &= \frac{t_2 - vx_2/c^2 - t_1 + vx_1/c^2}{\sqrt{1 - v^2/c^2}} \\ &= \frac{t_2 - t_1}{\sqrt{1 - v^2/c^2}} \quad \text{because } x_1 = x_2 \\ &= \frac{\Delta T_0}{\sqrt{1 - v^2/c^2}}, \end{aligned}$$

which is the correct expression for time dilation.

Derive length contraction from the Lorentz transformations:
A stick is at rest in the unprimed t, x coordinate system. Two events, #1 at (t_1, x_1) and #2 at (t_2, x_2), occur at different times $t_1 \neq t_2$ but at either end of the stick. And because the stick is at rest in the unprimed coordinate system, the *proper length* of the stick is

$$L_0 = x_2 - x_1$$

It has been arranged that these same two events occur at the same time in the primed coordinate system, $t_1' = t_2'$. So, in the primed coordinates the length of the stick is measured to be $L' = x'_2 - x'_1$. Now, the proper length of the stick is

$$\begin{aligned} L_0 &= x_2 - x_1 \\ &= \frac{x'_2 - vt'_2}{\sqrt{1 - v^2/c^2}} - \frac{x'_1 - vt'_1}{\sqrt{1 - v^2/c^2}} \\ &= \frac{x'_2 - vt'_2 - x'_1 + vt'_1}{\sqrt{1 - v^2/c^2}} \\ &= \frac{x'_2 - x'_1}{\sqrt{1 - v^2/c^2}} \quad \text{because } t'_1 = t'_2 \\ &= \frac{L'}{\sqrt{1 - v^2/c^2}} \quad \text{and, finally} \\ L' &= L_0\sqrt{1 - v^2/c^2} \end{aligned}$$

which is the correct expression for length contraction.

Consider the Galilean addition of velocities:
With the Lorentz transformations in hand, we can now see how velocities are viewed from different coordinate systems. First consider the Galilean addition of velocities. Consider a bird flying along, and note two nearby events along the bird's path—perhaps the events are two flaps of the bird's wings. These events are noted to occur at (t_1, x_1, y_1) and (t_2, x_2, y_2) in the unprimed coordinate system, and at (t'_1, x'_1, y'_1) and (t'_2, x'_2, y'_2) in the primed coordinate system. The components of the speed of the bird in the unprimed system are

$$u_x = \frac{x_2 - x_1}{t_2 - t_1} = \frac{\Delta x}{\Delta t} \qquad\qquad u_y = \frac{y_2 - y_1}{t_2 - t_1} = \frac{\Delta y}{\Delta t}$$

and in the primed coordinates

$$u'_x = \frac{x'_2 - x'_1}{t'_2 - t'_1} = \frac{\Delta x'}{\Delta t'} \qquad\qquad u'_y = \frac{y'_2 - y'_1}{t'_2 - t'_1} = \frac{\Delta y'}{\Delta t'}$$

Now use the Lorentz transformations to relate these different speeds:

$$\begin{aligned} u_x &= \frac{x_2 - x_1}{t_2 - t_1} = \frac{\Delta x}{\Delta t} \\ &= \frac{(\Delta x' + v\Delta t')/\sqrt{1 - v^2/c^2}}{(\Delta t' + v\Delta x'/c^2)/\sqrt{1 - v^2/c^2}} \\ &= \frac{\Delta x'/\Delta t' + v}{1 + v\Delta x'/\Delta t' c^2} \quad \text{and, finally} \\ u_x &= \frac{u'_x + v}{1 + v u'_x/c^2}, \end{aligned}$$

and

$$
\begin{aligned}
u_y &= \frac{y_2 - y_1}{t_2 - t_1} = \frac{\Delta y}{\Delta t} \\
&= \frac{\Delta y'}{(\Delta t' + v\Delta x'/c^2)/\sqrt{1 - v^2/c^2}} \\
&= \frac{\Delta y'/\Delta t'}{(1 + v\Delta x'/\Delta t' c^2)/\sqrt{1 - v^2/c^2}} \quad \text{and, finally} \\
u_y &= \frac{u'_y \sqrt{1 - v^2/c^2}}{1 + v u'_x/c^2}.
\end{aligned}
$$

If v and u'_x and u'_y are all less than c then u_x, u_y, and also $\sqrt{u_x^2 + u_y^2}$ are all less than c. Try this out with $v = u'_x = u'_y = 9c/10$.