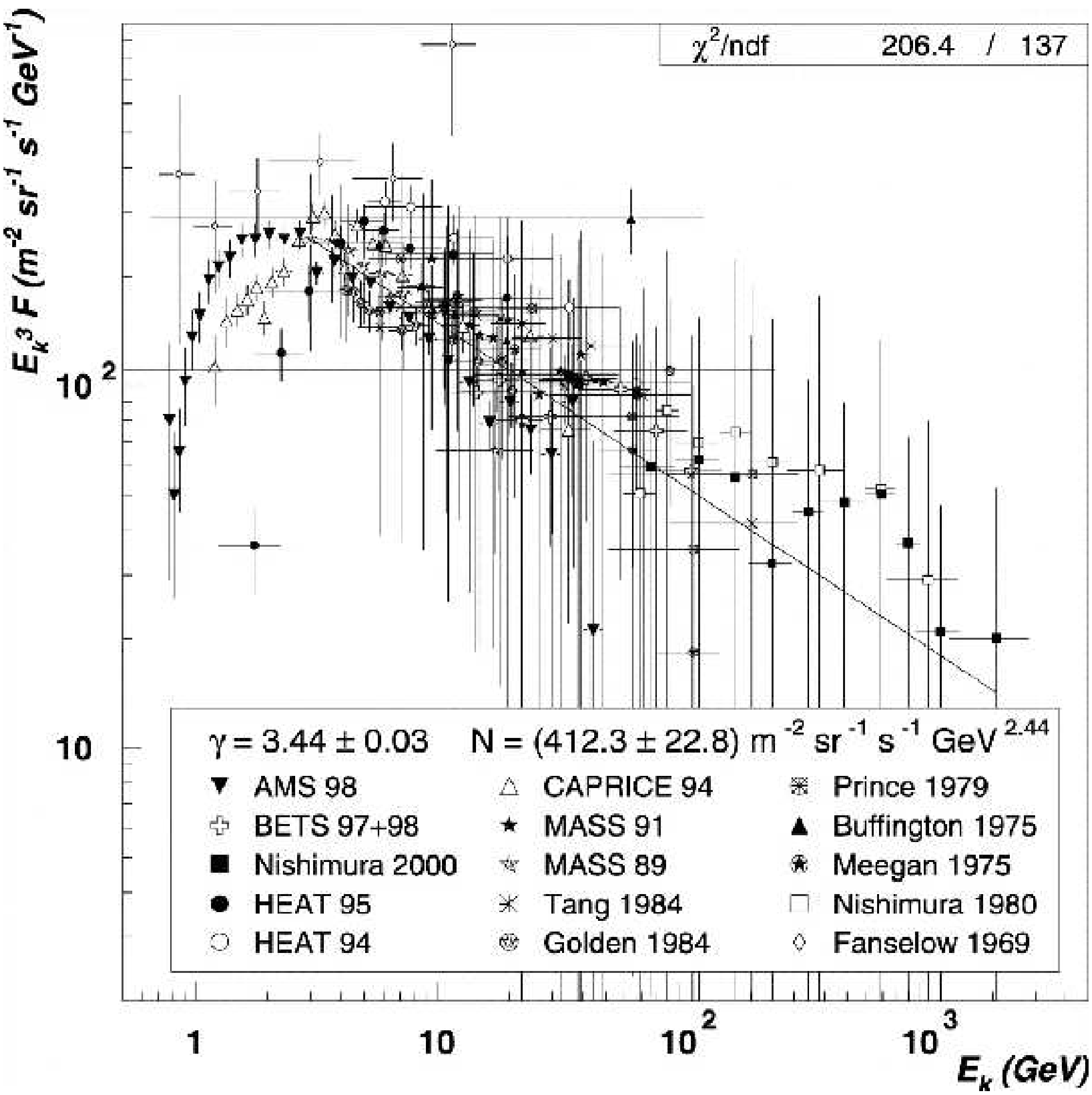
Synchrotron Sources

Spectra of Optically Thin Sources

If a synchrotron source containing any arbitrary distribution of electron energies is optically thin ($\tau \ll 1$), then its low-frequency spectrum is the superposition of the spectra from individual electrons and can never rise more rapidly than the $1/3$ power of frequency. In other words, the [negative] spectral index $\alpha \equiv -d\log P_\nu / d\log \nu$ (be careful not to confuse this α with the electron pitch angle α) must always be greater than $-1/3$. Most astrophysical sources of synchrotron radiation have spectral indices near $\alpha \approx 0.7$ at high frequencies where they are optically thin, and as we shall soon see, their overall spectral indices primarily reflect their electron energy distributions.



The energy spectrum of cosmic-ray electrons in the local interstellar medium (Casadei, D., & Bindi, V. 2004, ApJ, 612,262). In the energy range above a few GeV, N(E) is a power law with slope $\delta \approx 2.4$.

The observed energy distribution of cosmic-ray electrons in our Galaxy is roughly a power law:

$$\boxed{N(E)dE \approx KE^{-\delta}dE} \qquad (5D1)$$

where $N(E)dE$ is the number of electrons per unit volume with energies E to $E+dE$. The energy range around $\gamma \sim 10^4$ is relevant to the production of radio radiation, and there the power-law slope is $\delta \sim +2.4$. Because $N(E)$ is nearly a power law over more than two decades of energy and the critical frequency ν_c is proportional to energy squared, we expect the synchrotron spectrum to reflect this power law over a frequency range of at least $(10^2)^2 = 10^4$. Consequently, we can ignore the detailed spectra of individual electrons, which are smeared out in the observed spectrum by this broad power-law energy distribution. We make the very simple and crude approximation that each electron radiates all of its power

$$P = -\frac{dE}{dt} = \frac{4}{3}\sigma_{\rm T}\beta^2\gamma^2 c U_{\rm B}$$

at the single frequency

$$\nu \approx \gamma^2 \nu_{\rm G}$$

which is very close to the critical frequency. Then the emission coefficient of synchrotron radiation by an ensemble of electrons is

$$\epsilon_\nu d\nu = -\frac{dE}{dt} N(E) dE$$

where

$$E = \gamma m_{\rm e} c^2 \approx \left(\frac{\nu}{\nu_{\rm G}}\right)^{1/2} m_{\rm e} c^2 \ .$$

Differentiating E gives

$$\frac{dE}{d\nu} \approx \frac{m_{\rm e} c^2 \nu^{-1/2}}{2\nu_{\rm G}^{1/2}}$$

so

$$\epsilon_\nu \approx \left(\frac{4}{3}\sigma_{\rm T}\beta^2\gamma^2 c U_{\rm B}\right)(KE^{-\delta})\left(\frac{m_{\rm e} c^2 \nu^{-1/2}}{2\nu_{\rm G}^{1/2}}\right)$$

Eliminating E in favor of $\nu/\nu_{\rm G}$ and ignoring the physical constants in this equation for ϵ_ν results in the proportionality

$$\epsilon_\nu \propto \left(\frac{\nu}{\nu_{\rm G}}\right) B^2 \left(\frac{\nu}{\nu_{\rm G}}\right)^{-\delta/2} (\nu\,\nu_{\rm G})^{-1/2}$$

$$\epsilon_\nu \propto \left(\frac{\nu}{B}\right) B^2 \left(\frac{\nu}{B}\right)^{-\delta/2} (\nu\,B)^{-1/2}$$

since $\nu_{\rm G} \propto B$. We finally get:

$$\boxed{\epsilon_\nu \propto B^{(\delta+1)/2} \nu^{(1-\delta)/2}} \quad (5D2)$$

Since $\nu^{-\alpha} \propto \nu^{(1-\delta)/2}$,

$$\boxed{\alpha = \frac{\delta - 1}{2}} \quad (5D3)$$

That is, the synchrotron spectrum of a power-law energy distribution is itself a power law, and the equation above relates the slopes of these two power laws.

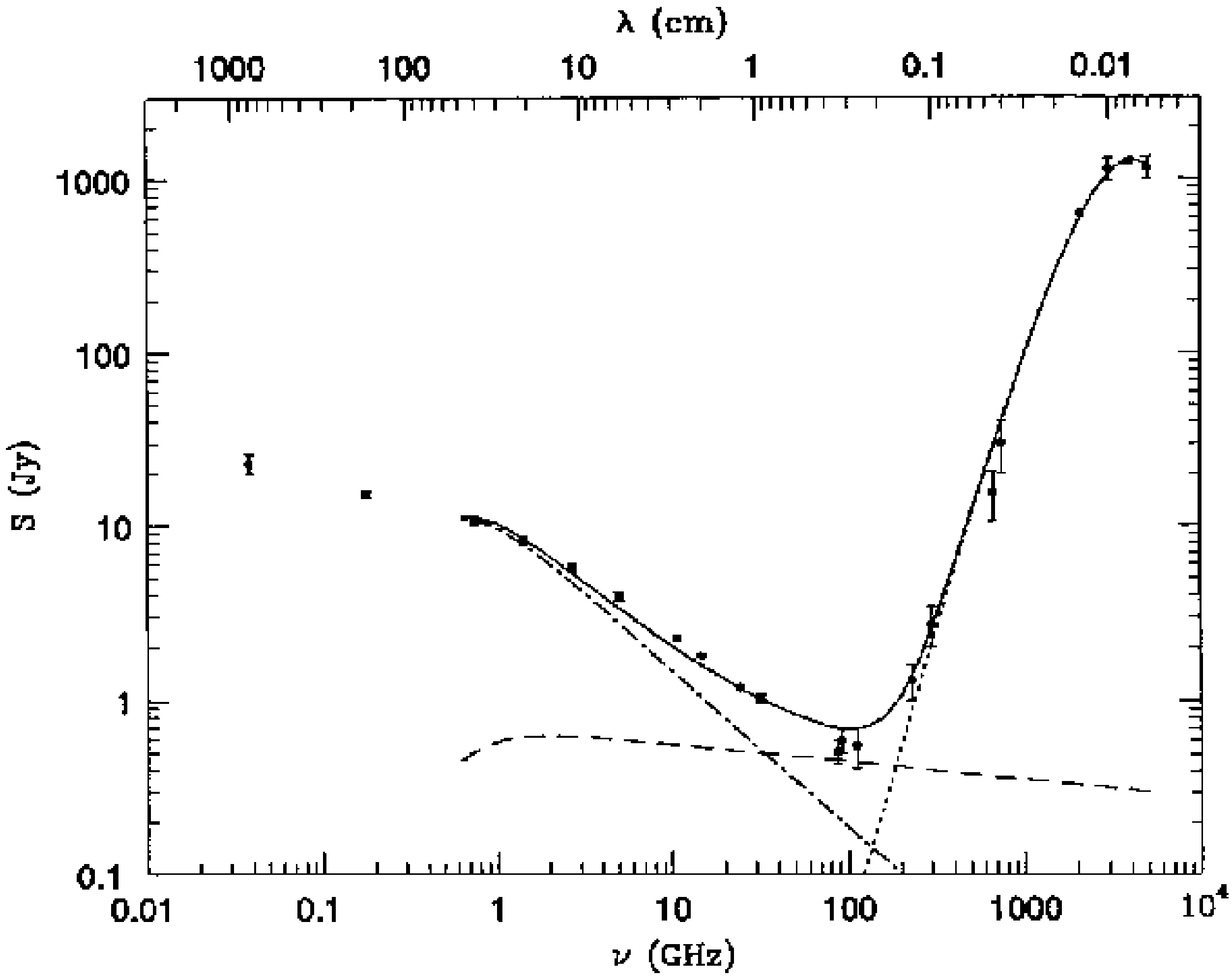
Example: In our Galaxy $\delta \approx 2.4$, so we expect

$$\epsilon_\nu \propto B^{1.7} \nu^{-0.7}$$

and hence the (negative) spectral index should be

$$\alpha \approx 0.7 \ ,$$

which is in agreement with observation. This is also the typical spectral index of most optically thin extragalactic radio sources, even radio galaxies and quasars. It reflects the power-law energy distribution of cosmic rays accelerated in shocks, the shocks produced by supernova remnants expanding into the ambient interstellar medium for example.



Synchrotron radiation (dot-dash line) from cosmic-ray electrons accelerated by the supernova remnants of relatively massive ($M > 8 M_\odot$) and short-lived ($T < 3 \times 10^7$ yr) stars dominates the radio continuum emission of the nearby starburst galaxy M82 at frequencies $\nu < 30$ GHz. Thermal emission (dashed line) from HII regions ionized primarily by even more massive ($M > 15 M_\odot$) and shorter-lived stars is strongest between about 30 and 200 GHz. At frequencies well below 1 GHz, free-free absorption flattens the overall spectrum.

Minimum Energy and Equipartition

What is the minimum energy required to produce a synchrotron source of a given luminosity? The existence of the source requires relativistic electrons with some energy density $U_{\rm e}$ and a magnetic field whose energy density is $U_{\rm B} = B^2/(8\pi)$.

To estimate $U_{\rm e}$, we assume a power-law electron energy distribution

$$N(E) \approx K E^{-\delta}$$

spanning the energy range $E_{\rm min}$ to $E_{\rm max}$ needed to produce synchrotron radiation over the observed frequency range $\nu_{\rm min}$ to $\nu_{\rm max}$. Then

$$\int^{E_{\rm max}}$$

$$U_{\rm e} = \int_{E_{\rm min}}^{E_{\rm max}} E N(E)\, dE$$

For a given synchrotron luminosity

$$L = \int_{\nu_{\rm min}}^{\nu_{\rm max}} L_\nu d\nu,$$

$$\frac{U_{\rm e}}{L} \propto \frac{\int_{E_{\rm min}}^{E_{\rm max}} E N(E)\, dE}{-\int_{E_{\rm min}}^{E_{\rm max}} (dE/dt) N(E)\, dE}$$

Substituting $N(E) = KE^{-\delta}$ and the synchrotron power emitted per electron $(-dE/dt) \propto B^2 E^2$ gives

$$\frac{U_{\rm e}}{L} \propto \frac{K \int_{E_{\rm min}}^{E_{\rm max}} E^{1-\delta} dE}{K B^2 \int_{E_{\rm min}}^{E_{\rm max}} E^{2-\delta} dE}$$

$$\frac{U_{\rm e}}{L} \propto \frac{E^{2-\delta}\Big|_{E_{\rm min}}^{E_{\rm max}}}{B^2 E^{3-\delta}\Big|_{E_{\rm min}}^{E_{\rm max}}}$$

Since electrons with energy E emit most of the radiation seen at frequency $\nu \propto E^2 B$, the electron energy needed to produce radiation at frequency ν scales as

$$E \propto B^{-1/2}$$

If we consider the energy content of only those electrons that emit in a fixed frequency range (e.g., from $\nu_{\rm min} \sim 10^7$ Hz to $\nu_{\rm max} \sim 10^{11}$ Hz), then the energy limits $E_{\rm min}$ and $E_{\rm max}$ are both proportional to $B^{-1/2}$ and

$$\frac{U_{\rm e}}{L} \propto \frac{(B^{-1/2})^{2-\delta}}{B^2 (B^{-1/2})^{3-\delta}} = \frac{B^{-1+\delta/2}}{B^2 B^{-3/2+\delta/2}} = B^{-3/2}$$

We conclude that

$$\boxed{U_{\rm e} \propto B^{-3/2}} \quad (5D4)$$

and we already know that

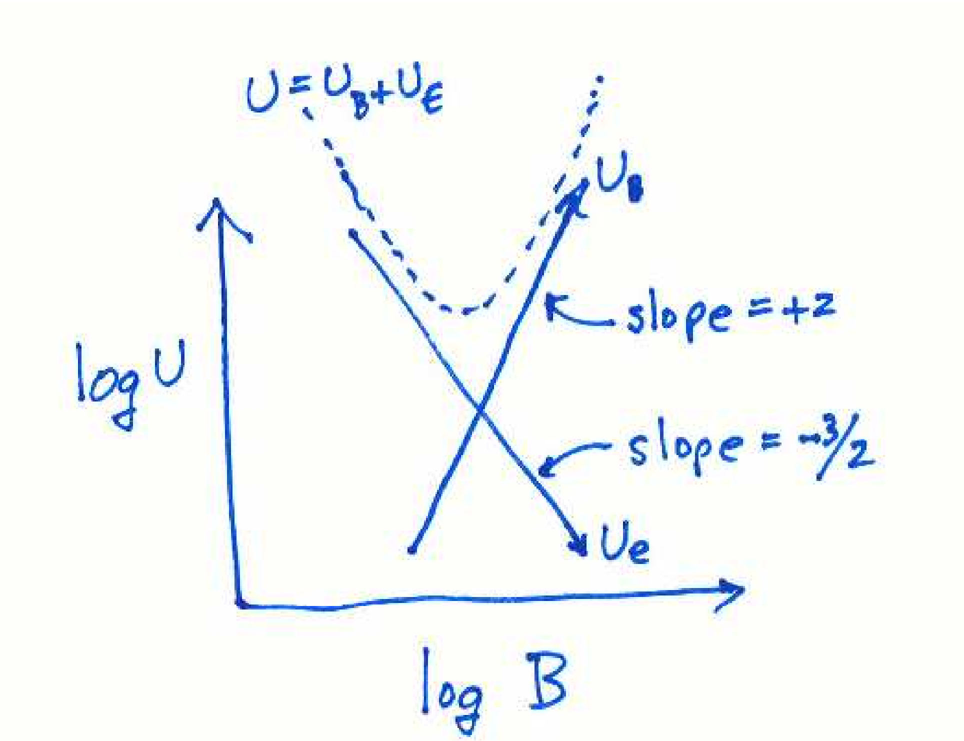
$$U_{\rm B} \propto B^2 \ .$$

The "invisible" cosmic-ray protons and heavier ions emit negligible synchrotron power but they still contribute to the total cosmic-ray particle energy. If we call the ion/electron energy ratio η, then the total energy density in cosmic rays is $(1+\eta)U_{\rm e}$. The total energy density U of both cosmic rays and magnetic fields is

$$\boxed{U = (1+\eta)U_{\rm e} + U_{\rm B}} \qquad (5D5)$$

We cannot measure η directly in distant radio sources, but cosmic rays collected near the Earth have $\eta \approx 40$.

The greatly differing dependences of $U_{\rm e}$ and $U_{\rm B}$ on B means that the total (cosmic ray plus magnetic) energy density $U(B)$ has a fairly sharp minimum near the point at which $(1+\eta)U_{\rm e} \approx U_{\rm B}$.



For a source of a given synchrotron luminosity, the particle energy density $U_{\rm E} \equiv (1+\eta)U_{\rm e}$ is proportional to $B^{-3/2}$ and the magnetic energy density $U_{\rm B}$ is proportional to B^2. The total energy density $U = U_{\rm E} + U_{\rm B}$ has a fairly sharp minimum near equipartition of the particle and magnetic energy densities ($U_{\rm E} \approx U_{\rm B}$).

The minimum of the total energy density U occurs at

$$\frac{dU}{dB} = \frac{d[(1+\eta)U_{\rm e} + U_{\rm B}]}{dB} = 0$$

First we evaluate the electron energy density.

$$dU \quad \left(3\right) \quad 3$$

$$\frac{dU_{\rm e}}{dB}\cdot U_{\rm e}^{-1} = -\left(\frac{3}{2}\right)B^{-5/2}B^{3/2} = -\frac{3}{2B}$$

so

$$\frac{dU_{\rm e}}{dB} = -\frac{3U_{\rm e}}{2B}$$

Next we evaluate the magnetic-field energy density.

$$\frac{dU_{\rm B}}{dB}\cdot U_{\rm B}^{-1} = \frac{2B}{B^2} = \frac{2}{B}$$

so

$$\frac{dU_{\rm B}}{dB} = \frac{2U_{\rm B}}{B}$$

Inserting these results into the minimum-energy equation gives

$$\frac{d[(1+\eta)U_{\rm e}]}{dB} + \frac{dU_{\rm B}}{dB} = 0 = -\frac{3(1+\eta)U_{\rm e}}{2B} + \frac{2U_{\rm B}}{B}$$

At minimum energy, the ratio of particle to field energy is

$$\boxed{\frac{\text{particle energy}}{\text{field energy}} = \frac{(1+\eta)U_{\rm e}}{U_{\rm B}} = \frac{4}{3}} \qquad \text{(5D6)}$$

This ratio is nearly unity. Thus ***minimum energy*** implies (near) ***equipartition*** of energy: the total cosmic-ray energy density (including the nonradiating ions) $(1+\eta)U_{\rm e}$ is nearly equal to the total magnetic energy density $U_{\rm B}$. We don't really know if equipartition exists in most sources, but radio astronomers often assume so, for several reasons:

(1) It is physically plausible—systems with interacting components often tend toward equipartition.

(2) Large and luminous extragalactic radio sources such as Cyg A have enormous energy requirements even near equipartition; the problem of explaining the large energy is even worse otherwise.

(3) It eliminates an unknown parameter and permits estimates of the relativistic particle energies and the magnetic field strengths of radio sources.

Getting the actual numerical values of the particle and magnetic field energies from the synchrotron emission coefficient is a straightforward by tedious algebraic chore (see Rohlfs & Wilson Section 9.10). Below are the results (from Pacholczyk's *Radio Astrophysics*, p. 171).

For a spherical radio source with radius R and magnetic field strength B, the total magnetic

For a spherical radio source with radius R and magnetic field strength B, the total magnetic energy is

$$E_{\rm B} = U_{\rm B} V = \frac{B^2}{8\pi}\frac{4\pi R^3}{3} = \frac{B^2 R^3}{6}$$

$$B_{\rm min} = [4.5(1+\eta)c_{12}L]^{2/7} R^{-6/7} \text{ Gauss}$$

$$E_{\rm min}(\text{total}) = c_{13}[(1+\eta)L]^{4/7} R^{9/7} \text{ ergs}$$

where the radio luminosity L is conventionally integrated over the observable frequency range $\nu = 10^7$ Hz to $\nu = 10^{11}$ Hz,

$$L = \int_{\nu_{\rm min}=10^7 \text{ Hz}}^{\nu_{\rm max}=10^{11} \text{ Hz}} L_\nu d\nu \ , \qquad L_\nu = 4\pi D^2 S_\nu \ ,$$

D is the source distance, S_ν is its flux density at frequency ν, and

$$1+\eta \equiv \frac{\text{energy in all relativistic particles}}{\text{energy in relativistic electrons}} \ .$$

The minimum total energy in relativistic particles and fields occurs when

$$\frac{(1+\eta)U_{\rm e}}{U_{\rm B}} \approx \frac{4}{1+\delta},$$

where

$$\delta = 2\alpha + 1 \sim 2.4 \ .$$

The ***synchrotron lifetime*** of a source is defined as the ratio of electron energy to the energy loss rate L from synchrotron radiation:

$$\tau \equiv \frac{E_{\rm e}}{L} \ .$$

$$\tau \approx c_{12} B_\perp^{-3/2}$$

The functions c_{12} and c_{13} in Gaussian cgs units are listed in Table 8 of Pacholczyk's *Radio Astrophysics* reproduced below.

The functions:†

$$c_{12} = c_2^{-1} c_1^{1/2} \frac{2\alpha - 2}{2\alpha - 1} \cdot \frac{\nu_1^{(1-2\alpha)/2} - \nu_2^{(1-2\alpha)/2}}{\nu_1^{1-\alpha} - \nu_2^{1-\alpha}}$$

$$c_{13} = 0.921 \cdot c_{12}^{4/7}$$

for $\nu_1 = 10^7$ Hz and $\nu_2 = 10^{10}$ and 10^{11} Hz.

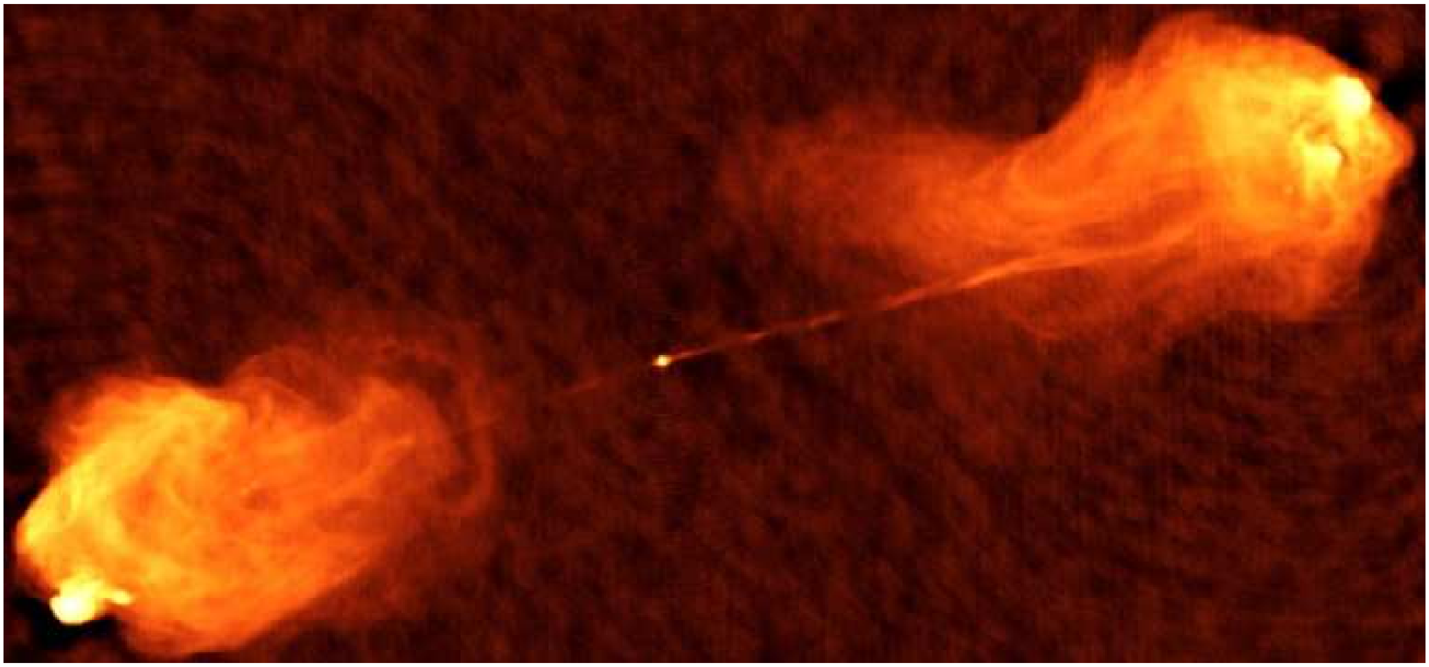
α	$\nu_2 = 10^{10}$ Hz		$\nu_2 = 10^{11}$ Hz	
	c_{12}	c_{13}	c_{12}	c_{13}
0.2	2.5 *E* 07	1.6 *E* 04	8.3 *E* 06	8.3 *E* 03
0.3	2.8 *E* 07	1.7 *E* 04	9.8 *E* 06	9.1 *E* 03
0.4	3.2 *E* 07	1.8 *E* 04	1.2 *E* 07	1.0 *E* 04
0.5	3.7 *E* 07	2.0 *E* 04	1.6 *E* 07	1.2 *E* 04
0.6	4.5 *E* 07	2.2 *E* 04	2.0 *E* 07	1.4 *E* 04
0.7	5.4 *E* 07	2.5 *E* 04	2.8 *E* 07	1.7 *E* 04
0.8	6.5 *E* 07	2.7 *E* 04	3.9 *E* 07	2.0 *E* 04
0.9	7.8 *E* 07	3.0 *E* 04	5.4 *E* 07	2.4 *E* 04
1.0	9.3 *E* 07	3.3 *E* 04	7.1 *E* 07	2.8 *E* 04
1.1	1.1 *E* 08	3.6 *E* 04	9.3 *E* 07	3.3 *E* 04
1.2	1.3 *E* 08	4.0 *E* 04	1.1 *E* 08	3.7 *E* 04

† For $\alpha = 1/2$ and 1 the functions c_{12} and c_{13} have values following from the appropriate formulae resulting from the integration of equations (7.4) and (7.5).

Table 8 from Pacholczyk's Radio Astrophysics. *Here $\nu_1 = \nu_{\rm min}$ and $\nu_2 = \nu_{\rm max}$.*

Example: What are the minimum-energy magnetic field strength and the minimum total energy of Cygnus A, a luminous double radio source (see the VLA image below) at distance $D \approx 230$ Mpc (for $H_0 = 75$ km s^{-1} Mpc^{-1})? The lobe radii are $R \approx 30$ kpc and the total flux density of Cyg A is

$$S_\nu \approx 2000 \text{ Jy } \left(\frac{\nu}{\rm GHz} \right)^{-0.8}$$



First we convert the data from "astronomical" units to cgs units:

$$R = 30\ \mathrm{kpc} \times \frac{10^3\ \mathrm{pc}}{\mathrm{kpc}} \times \frac{3.09 \times 10^{18}\ \mathrm{cm}}{\mathrm{pc}} \approx 9.0 \times 10^{22}\ \mathrm{cm}$$

$$S_\nu = 2000\ \mathrm{Jy} \left(\frac{10^{-23}\ \mathrm{erg\ s^{-1}\ Hz^{-1}\ cm^{-2}}}{\mathrm{Jy}} \right) \left(\frac{\nu}{10^9\ \mathrm{Hz}} \right)^{-0.8}$$

$$S_\nu = \frac{3.17 \times 10^{-13}\ \mathrm{erg\ s^{-1}\ Hz^{-1}}}{\mathrm{cm}^2} \times \left(\frac{\nu}{\mathrm{Hz}} \right)^{-0.8}$$

$$D = 230\ \mathrm{Mpc} \times \frac{10^6\ \mathrm{pc}}{\mathrm{Mpc}} \times \frac{3.09 \times 10^{18}\ \mathrm{cm}}{\mathrm{pc}} \approx 7.1 \times 10^{26}\ \mathrm{cm}$$

The spectral luminosity of Cyg A is

$$L_\nu \approx 4\pi D^2 S_\nu = 4\pi (7.1 \times 10^{26}\ \mathrm{cm})^2 \times \frac{3.17 \times 10^{-13}\ \mathrm{erg\ s^{-1}\ Hz^{-1}}}{\mathrm{cm}^2} \times \left(\frac{\nu}{\mathrm{Hz}} \right)^{-0.8}$$

$$L_\nu \approx 2.0 \times 10^{42}\ \mathrm{erg\ s^{-1}\ Hz^{-1}} \times \left(\frac{\nu}{\mathrm{Hz}} \right)^{-0.8}$$

The total radio luminosity of Cyg A in the frequency range 10^7 Hz to 10^{11} Hz is:

$$L = \int_{10^7\ \mathrm{Hz}}^{10^{11}\ \mathrm{Hz}} L_\nu d\nu$$

$$L \approx 2.0 \times 10^{42} \text{ erg s}^{-1} \text{ Hz}^{-1} \left(\frac{\nu^{0.2}}{0.2} \right) \Bigg|_{10^{7} \text{ Hz}}^{10^{11} \text{ Hz}}$$

$$L \approx 2.0 \times 10^{42} \text{ erg s}^{-1} \left[\frac{(10^{11})^{0.2} - (10^{7})^{0.2}}{0.2} \right]$$

$$L \approx 1.33 \times 10^{45} \text{ erg s}^{-1}$$

In units of the *bolometric* (jargon for "as observed by a bolometer" and meaning integrated over *all* frequencies) solar luminosity $L_\odot \approx 3.83 \times 10^{33}$ erg s^{-1}, the *radio* luminosity of Cyg A is

$$\frac{L}{L_\odot} \approx \frac{1.33 \times 10^{45} \text{ erg s}^{-1}}{3.83 \times 10^{33} \text{ erg s}^{-1}} \approx 3.5 \times 10^{11}$$

Thus the radio power from Cyg A exceeds the bolometric output from a galaxy of stars similar to our Galaxy. The energy appears to originate in a compact object at the center of the host galaxy. How massive must this compact object be to produce such a luminous source?

Eddington Limit

What is the maximum luminosity of an astronomical object of total mass M in a steady state? The outward radiation pressure cannot exceed gravity. For example, radiation pressure would expel the outer layers of a star in the form of a wind, or accretion onto a compact object would be disrupted. Even if the atmosphere or infalling material is ionized hydrogen, the free electrons will Thomson-scatter outflowing radiation. Each electron being blown away by radiation pressure will drag along one proton ($m_{\rm p} \gg m_{\rm e}$) to maintain charge neutrality. Balancing the radiation and gravitational forces on each electron/proton pair at distance r from the accreting object gives the ***Eddington Luminosity***:

$$\frac{L_{\rm E}}{4\pi r^2 c} \sigma_{\rm T} = \frac{GM(m_{\rm p} + m_{\rm e})}{r^2} \approx \frac{GMm_{\rm p}}{r^2}$$

Note that the distance r drops out and

$$L_{\rm E} \approx \frac{4\pi G M m_{\rm p} c}{\sigma_{\rm T}}$$

In cgs units

$$L_{\rm E}(\text{erg s}^{-1}) = \frac{4\pi \times 6.67 \times 10^{-8} \text{ dyne cm}^2 \text{ g}^{-2} \times M \times 1.66 \times 10^{-24} \text{ g} \times 3 \times 10^{10} \text{ cm s}^{-1}}{6.65 \times 10^{-25} \text{ cm}^2}$$

$$L_{\rm E}({\rm erg~s^{-1}}) = 6.28 \times 10^4 M({\rm g})$$

Normalized to "solar" units $L_\odot \approx 3.83 \times 10^{33}$ erg s^{-1} and $M_\odot \approx 1.99 \times 10^{33}$ g,

$$\left(\frac{L_{\rm E}}{L_\odot}\right) \approx \frac{6.28 \times 10^4 \times 1.99 \times 10^{33}~{\rm g}}{3.83 \times 10^{33}~{\rm erg~s^{-1}}} \left(\frac{M}{M_\odot}\right)$$

$$\boxed{\left(\frac{L_{\rm E}}{L_\odot}\right) \approx 3.3 \times 10^4 \left(\frac{M}{M_\odot}\right)} \quad (5D7)$$

As the mass of a main-sequence star approaches $M \approx 100 M_\odot$, its luminosity approaches its Eddington luminosity. Very massive stars often have radiation-driven winds, and stable stars more massive than $100 M_\odot$ may not be possible.

Example: What does the Eddington limit give for the minimum mass for a source of luminosity $L \approx 3.5 \times 10^{11} L_\odot$? We make the *assumption* that the average luminosity of the central mass was at least this much at some times, so

$$\left(\frac{M}{M_\odot}\right) \geq \frac{3.5 \times 10^{11}}{3.3 \times 10^4} \approx 10^7$$

Note that the Eddington mass limit depends only on the instantaneous power emitted by the source, not on the total energy of the source, the source age, or any other indicator of its history.

Temperatures of Eddington-limited accretion disks near black holes

The Eddington limit is directly applicable to quasars and other sources having having luminous accretion disks. If the gravitational energy released by accretion is thermalized, the hottest and hence brightest material will be concentrated just outside the innermost stable orbit surrounding the central rotating black hole. The radius of this orbit is

$$r = 3 r_{\rm g} = 3 \times \frac{2GM}{c^2} ,$$

where $r_g = 2GM/c^2$ is the gravitational radius or ***Schwarzschild radius***. If the compact object is accreting enough matter to approach its Eddington luminosity, the combination of luminosity and radius determines the blackbody temperature T of the inner accretion disk.

$$L \approx 4\pi r^2 \sigma T^4 \approx L_{\rm E}$$

$$\left(\frac{6GM}{}\right)^2 \qquad \left(\frac{GMm_{\rm p}c}{}\right)$$

$$4\pi\left(\frac{6GM}{c^2}\right)^2\sigma T^4 \approx 4\pi\left(\frac{GMm_{\rm p}c}{\sigma_{\rm T}}\right)$$

$$T^4 \approx \left(\frac{m_{\rm p}c^5}{36G\sigma\sigma_{\rm T}}\right)M^{-1}$$

The more massive the black hole, the *cooler*. Inserting cgs values for the constants gives

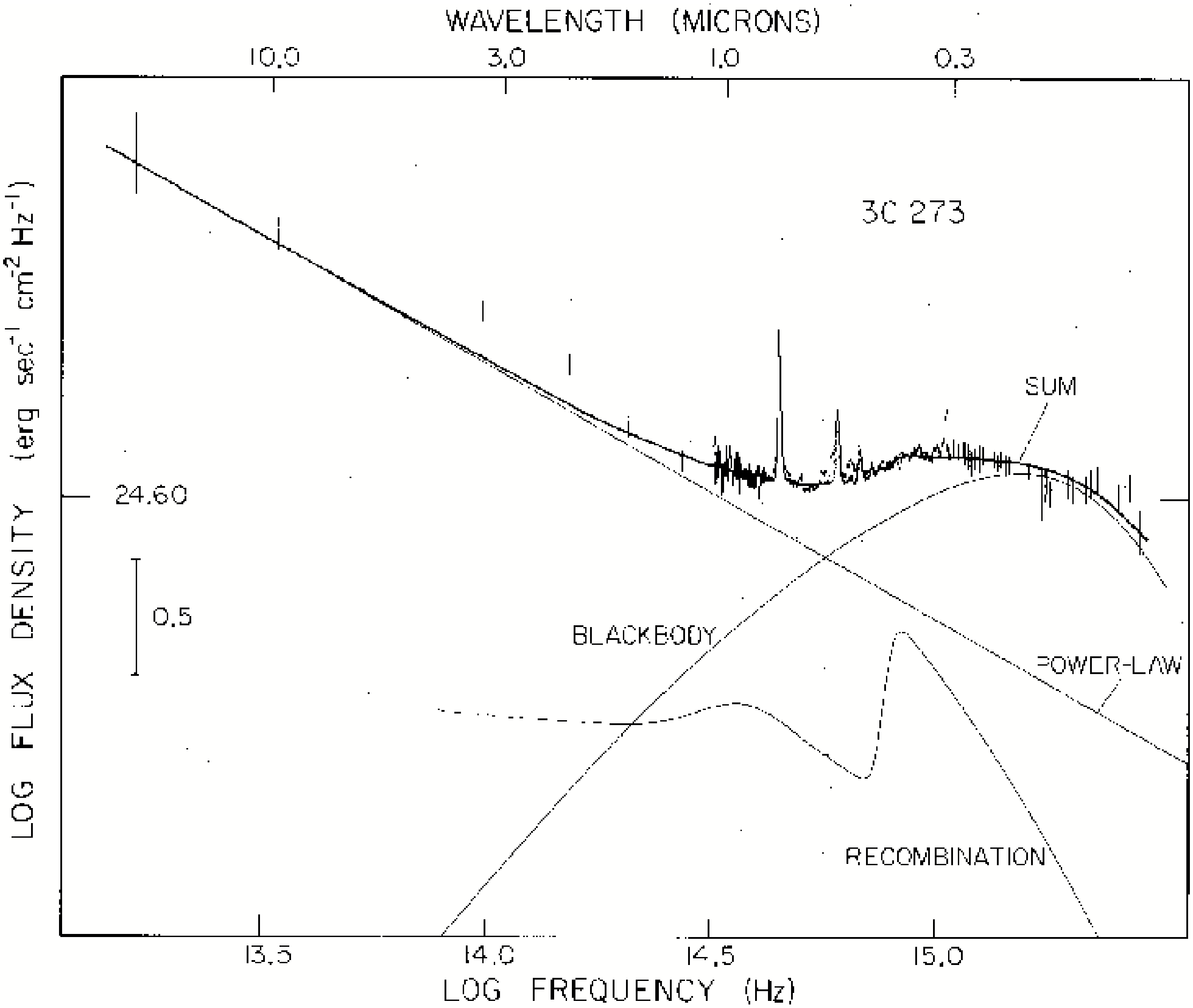
$$T^4 \approx \left(\frac{1.66\times10^{-24}\ {\rm g}\,(3\times10^{10}\ {\rm cm\ s^{-1}})^5}{36\times6.67\times10^{-8}\ {\rm erg\ cm^{-2}s^{-1}K^{-4}}\times6.65\times10^{-25}\ {\rm cm}^2}\right)M^{-1}$$

$$T^4 \approx 4.46\times10^{62}\ {\rm g\ K}^4 M^{-1}$$

In units of $M_\odot \approx 1.99\times10^{33}$ g,

$$\left(\frac{T}{\rm K}\right) \approx 2.2\times10^7\left(\frac{M_\odot}{M}\right)^{1/4}$$

Example: The spectrum of the quasar 3C 273 is the superposition of a power-law from synchrotron radiation and a thermal "big blue bump" peaking at ultraviolet wavelengths.



The spectrum of 3C273 (Malkom, M. A., & Sargent, W. L. W. 1982, ApJ, 254, 22).

If $M \sim 10^9 M_\odot$ then $T \sim 10^5$ K, in agreement with the observed thermal spectrum and accounting for the strong emission-line spectrum of ionized hydrogen. Many quasars have similar blue bumps and appear to be accreting at rates approaching the Eddington limit. Note that black holes with masses of only a few $M_\odot$ accreting at the Eddington limit will have much higher temperatures $T \sim 10^7$ K and be strong thermal X-ray sources.

Returning to Cyg A, we estimate the magnetic field strength $B_{\rm min}$ that minimizes the total energy in relativistic particles and magnetic fields. We approximate Cyg A by two equal lobes of radius $R \approx 30$ kpc and luminosity $L/2$, where L is the radio luminosity of the whole source.

$$B_{\rm min} \approx [4.5(1+\eta)c_{12}(L/2)]^{2/7} R^{-6/7}$$

$$B_{\rm min} \approx (4.5 \times 3.9 \times 10^7 \times 1.33 \times 10^{45} \text{ erg s}^{-1}/2)^{2/7} (9 \times 10^{22} \text{ cm})^{-6/7} (1+\eta)^{2/7}$$

The value of η is poorly constrained in extragalactic radio sources such as Cyg A. The cosmic rays accelerated by a supermassive black hole might be primarily electrons and positrons. Electrons and positrons have the same mass and charge (except for sign), so they are equally efficient at emitting synchrotron radiation and $\eta \approx 1$. If electrons and protons are acclerated to the same emitting synchrotron radiation and $\eta \approx 1$. If electrons and protons are acclerated to the same velocities (same γ), then the protons carry $m_{\rm p}/m_{\rm e} \sim 2 \times 10^3$ as much energy but emit almost nothing and $\eta \sim 2 \times 10^3$. Fortunately, $B_{\rm min} \propto \eta^{2/7}$ is only weakly dependent on η. Varying η from 1 to 2×10^3 only changes $(1+\eta)^{2/7}$ from about 1 to 9.

$$B_{\rm min} \approx 1.45 \times 10^{15} \times 2.1 \times 10^{-20} \times (1 \text{ to } 9) \text{ Gauss}$$

$$B_{\rm min} \approx (30 \text{ to } 300) \times 10^{-6} \sim 10^{-4} \text{ Gauss}$$

The minimum total energy of Cyg A is twice the energy of each lobe:

$$E_{\rm min} \approx 2(\text{lobes}) \times c_{13}[(1+\eta)L]^{4/7} R^{9/7}$$

$$E_{\rm min} \approx 2 \times 2.0 \times 10^4 \left(\frac{1.33 \times 10^{45} \text{ erg s}^{-1}}{2} \right)^{4/7} (9 \times 10^{22} \text{ cm})^{9/7} \times (1+\eta)^{4/7} ,$$

where $(1+\eta)^{4/7}$ is in the range of about 1 to 80.

$$E_{\rm min} \approx 4 \times 10^4 \times 4.1 \times 10^{25} \times 3.26 \times 10^{29} \times (1 \text{ to } 80) \text{ ergs}$$

$$E_{\rm min} \approx 5.4 \times 10^{59} \times (1 \text{ to } 80) \text{ ergs} \sim 5 \times 10^{60} \text{ ergs}$$

This enormous energy can be used to set another lower limit to the mass of the central object powering the radio source. If mass could be converted to energy with 100% efficiency, the minimum mass needed to produce $E_{\rm min}$ would be

$$M \geq \frac{E_{\rm min}}{c^2} \approx \frac{5 \times 10^{60} \text{ ergs}}{(3 \times 10^{10} \text{ cm s}^{-1})^2} \approx 6 \times 10^{39} \text{ g}$$

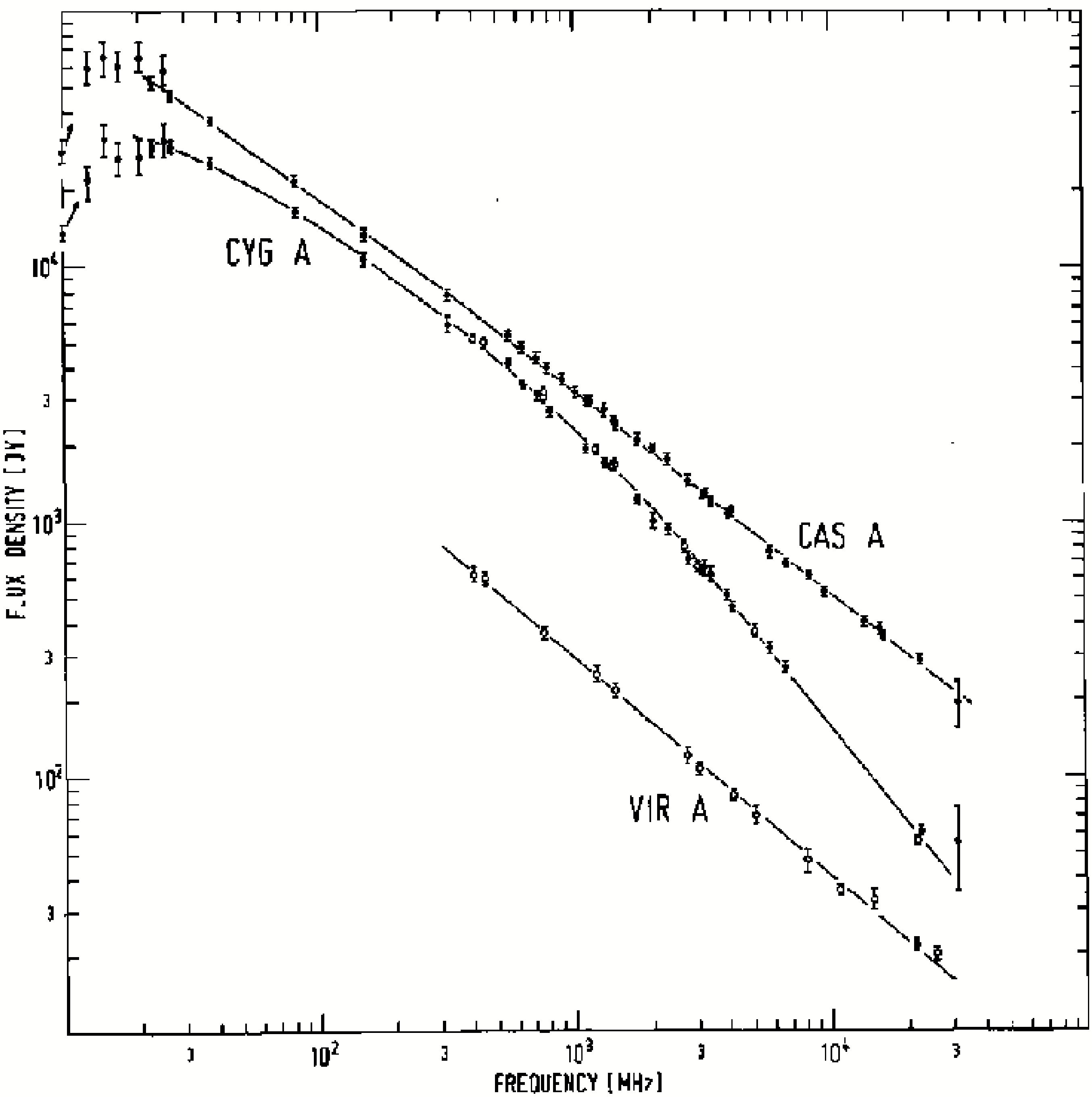
$$M \geq 6 \times 10^{39} \text{ g} \left(\frac{M_\odot}{1.99 \times 10^{33} \text{ g}} \right) \approx 3 \times 10^6 M_\odot$$

This is a very conservative lower limit. Nuclear fusion can only convert mass to energy with about 1% efficiency, so $M > 3 \times 10^8 M_\odot$ if the energy source were nuclear fusion. Accretion onto a spinning black hole can result in efficiencies up to $(1 - 3^{-1/2}) \approx 0.4$ in theory, so it is consistent with $M > 10^7 M_\odot$. In the literature, it is often assumed that mass is converted to energy with about 10% efficiency; this yields $M > 3 \times 10^7 M_\odot$. The small size of the radio core implied by Very Long Baseline Interferometery (VLBI) and variability of the core flux on time scales of months to years combined with the large minimum masses estimated from the Eddington limit and the total energy of the radio lobes together imply that the compact, massive object powering the radio source is a supermassive black hole. The adjective *supermassive* is used to indicate black holes much more massive than the most massive stars, $\sim 100 M_\odot$.

A lower limit to the age of the radio source Cyg A is the synchrotron lifetime of the relativistic electrons estimated by taking the ratio of the electron energy to the observed synchrotron luminosity:

$$\tau \geq \frac{E_{\min}/(1+\eta)}{L} \approx \frac{5.4 \times 10^{59} \text{ erg}\,(1+\eta)^{4/7}}{1.33 \times 10^{45} \text{ erg s}^{-1}(1+\eta)} \approx 4 \times 10^{14} \text{ s} \times \eta^{-3/7} \sim 10^{14} \text{ s} \sim 3 \times 10^{6} \text{ yr}$$

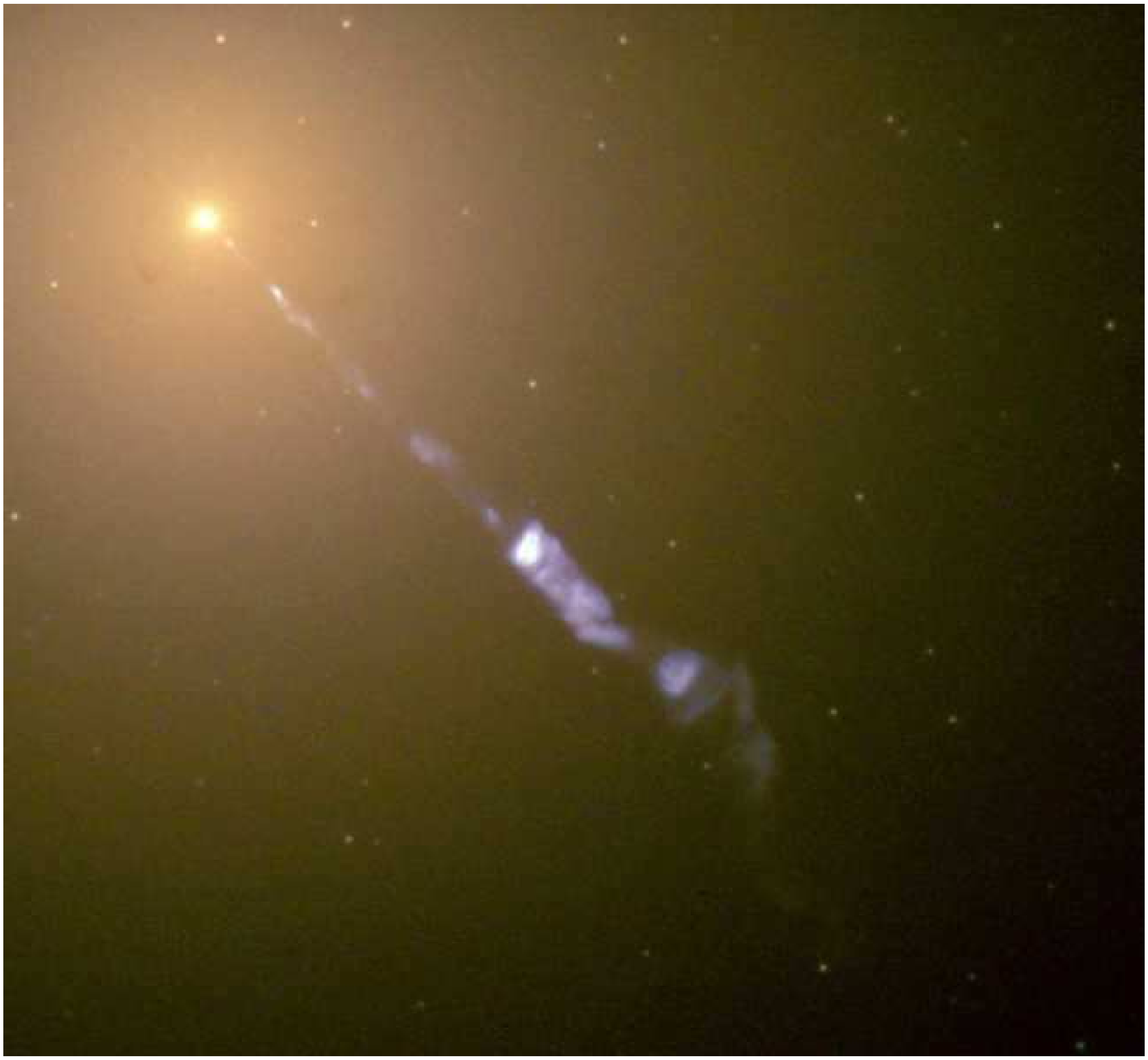
Since each electron radiates energy at a rate proportional to E^2 and the critical frequency is proportional to E^2, the most energetic electrons emitting at the highest frequencies have the shortest lifetimes. The rapid depletion of high-energy electrons causes the emitted radio spectrum to steepen at high frequencies.



The radio spectrum of Cyg A (and Cas A, Vir A) from Baars, J. W. M. et al. 1977, A&A, 61, 99. Note the spectral steepening above $\nu \sim 10^3$ MHz.

Suppose that new relativistic electrons are continously injected with a power-law energy distribution $N(E) \propto E^{-\delta_0}$ into a radio source. After a long time, electrons emitting at frequencies higher than ν will have been depleted by radiative losses $\propto E^2$, so these high-energy electrons will eventually have an energy distribution $N(E) \propto E^{-(\delta_0+1)}$. Consequently, the (negative) spectral index will be $\alpha_0 = (\delta_0 - 1)/2$ at low frequencies and approach $\alpha = (\delta_0 + 1 - 1)/2 = (\alpha_0 + 1/2)$ at higher frequencies; the high-frequency spectrum steepens by $\Delta\alpha = 1/2$.

If the observed cutoff frequency ν is very high, the synchrotron lifetime of electrons with $\nu_c \sim \nu$ may be less than the time needed for new relativistic electrons to travel from the core to the emitting feature in a jet or lobe. This implies *in situ acceleration*—something outside the core (e.g., shocks in the jet) must replenish the supply of relativistic electrons.



Optical synchrotron emission in the radio jet of Virgo A = M87. Image credit

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Synchrotron Self-Absorption

For every emission process there is an associated absorption process. The emitting particles in a source in local thermodynamic equilibrium (LTE) have a Maxwellian energy distribution, and such a source is called a *thermal source*. If the particle temperature is T, the source cannot have a brightness temperature greater than T. If the energy distribution of relativistic electrons in a synchrotron source were a (relativistic) Maxwellian, those electrons would have a characteristic temperature $T \sim E/3k$, and ***synchrotron self-absorption*** would prevent the brightness temperature from exceeding T. Astrophysical synchrotron sources are often called ***nonthermal sources*** because the energy distribution of the relativistic electrons is a power law and there is no single electron temperature T. However, self-absorption occurs regardless of the energy distribution.

In the approximation that electrons with energy $E = \gamma m_e c^2$ in a magnetic field of strength B emit only at the critical frequency

$$\nu_c \sim \frac{\gamma^2 e B}{2\pi m_e c} ,$$

the Lorentz factor γ of electrons emitting at frequency ν is:

$$\gamma \approx \left(\frac{2\pi m_e c \nu}{eB}\right)^{1/2} .$$

Since only electrons of one particular energy contribute to the emission and absorption at any one frequency in that approximation, the other electrons *could* have a relativistic Maxwellian energy distribution to match without changing the resulting emission and absorption at that frequency. Thus we expect that a sufficiently bright synchrotron source *will* be optically thick, and the brightness temperature at any frequency cannot exceed the effective temperature of those electrons emitting at that frequency.

In an ultrarelativistic gas, the ratio of specific heats at constant pressure and at constant volume is $c_p/c_v = 4/3$, not the nonrelativistic $5/3$, so the relation between electron energy E and temperature T_e is

$$E = 3kT_e \ , \ \text{not} \ (3/2)kT_e \ .$$

Thus the ***effective temperature*** of a relativistic electron is

$$T_e = \frac{E}{3k} = \frac{\gamma m_e c^2}{3k}$$

Eliminating γ in favor of ν gives the effective temperature of those electrons accounting for most of the radiation at frequency ν:

$$T_e \approx \left(\frac{2\pi m_e c \nu}{eB}\right)^{1/2} \frac{m_e c^2}{3k}$$

Numerically,

$$\left(\frac{T_e}{\mathrm{K}}\right) \approx 1.18 \times 10^6 \left(\frac{\nu}{\mathrm{Hz}}\right)^{1/2} \left(\frac{B}{\mathrm{Gauss}}\right)^{-1/2} \qquad (5D8)$$

br>

Example: What is the effective temperature of the relativistic electrons emitting synchrotron radiation at $\nu = 0.1\ \mathrm{GHz} = 10^8$ Hz if $B = 100\,\mu$Gauss $= 10^{-4}$ Gauss?

$$\left(\frac{T_e}{\mathrm{K}}\right) \approx 1.18 \times 10^6 \times (10^8)^{1/2} (10^{-4})^{-1/2} \approx 10^{12}$$

At a sufficiently low frequency ν, the brightness temperature T_b of any synchrotron source will approach the effective electron temperature T_e at that frequency and the source will become opaque. Starting with the definition of T_b:

$$I_\nu = \frac{2kT_b\nu^2}{c^2}$$

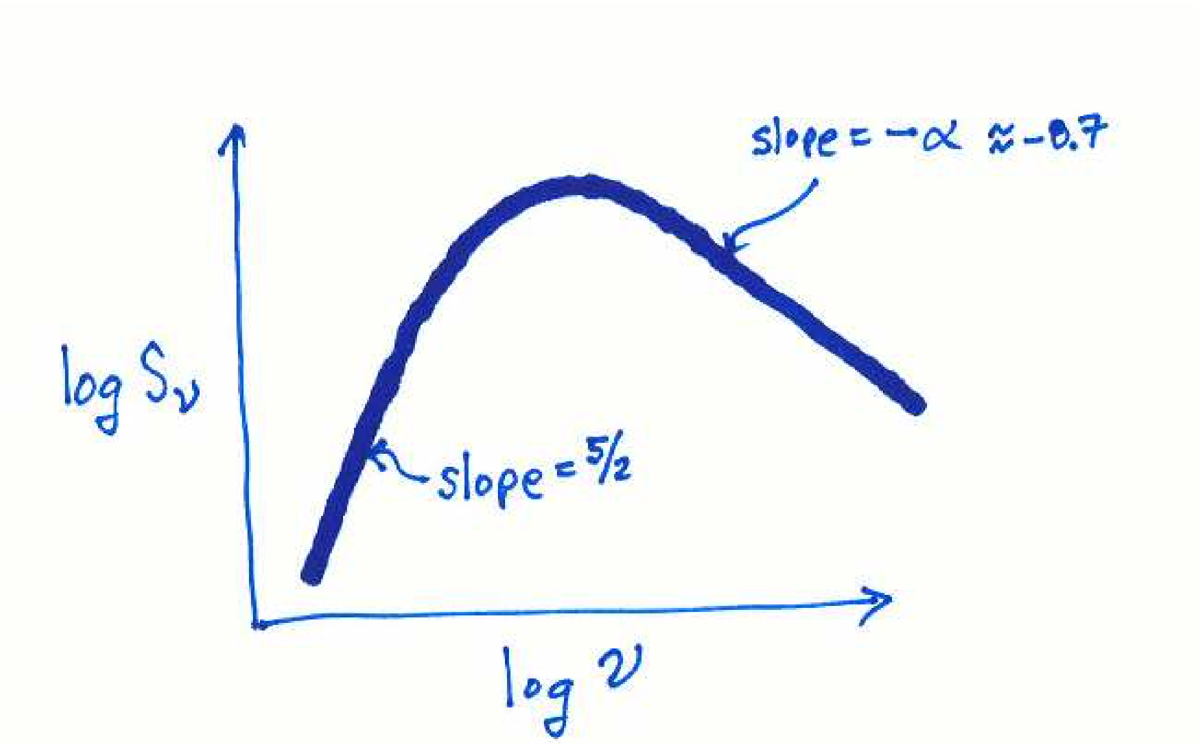
and setting $T_b \approx T_e$ gives

$$I_\nu \approx \frac{2kT_e\nu^2}{c^2} \propto \nu^{1/2}\nu^2 B^{-1/2}$$

Then, because flux density is proportional to I_ν for a source subtending a given solid angle Ω, the spectrum of a synchrotron self-absorbed and spatially homogeneous source is a power law of slope $5/2$:

$$S(\nu) \propto \nu^{5/2} \qquad (5D9)$$

independent of the slope δ of the electron-energy spectrum. The flux density of an opaque but truly thermal source (e.g., an HII region) is proportional to ν^2; the extra $\nu^{1/2}$ for synchrotron radiation comes from the fact that $T_e \propto \nu^{1/2}$.



The spectrum of an ideal homogeneous synchrotron source is a power law with slope $-\alpha = 5/2$ *at low frequencies where* $\tau \gg 1$. *Astrophysical sources are inhomogeneous, so their actual low-frequency spectral slopes are smaller than 5/2.*

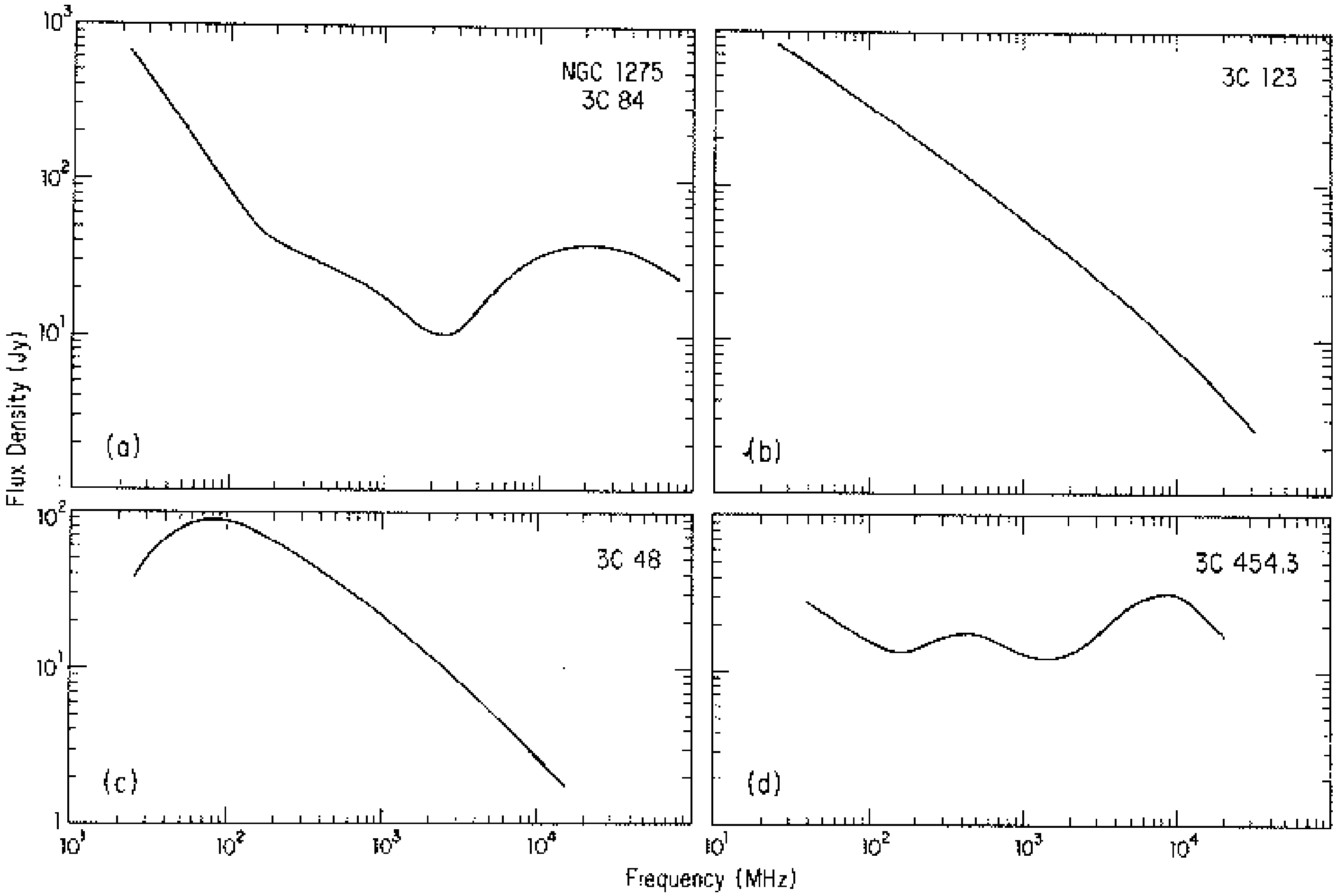
We can invert the equation for T_e to estimate the ***magnetic field strength*** in a self-absorbed source whose brightness temperature has been measured.

$$\left(\frac{B}{\text{Gauss}}\right) \approx 1.4 \times 10^{12} \left(\frac{\nu}{\text{Hz}}\right) \left(\frac{T_b}{\text{K}}\right)^{-2} \qquad (5D10)$$

Example: If a self-absorbed radio source is observed to have $T_b \approx 10^{11}$ K at $\nu = 1$ GHz, the magnetic field strength is

$$\left(\frac{B}{\text{Gauss}}\right) \approx 1.4 \times 10^{12} \times 10^9 \times (10^{11})^{-2} \approx 0.1$$

The spectra of "real" radio sources reflect the idealized spectra of uniform sources, but they are more complex because real sources have nonuniform magnetic fields and electron energy distributions in geometrically complex structures. Representative spectra of powerful radio galaxies and quasars are illustrated below.



Spectra of radio galaxies and quasars. The radio source 3C 84 in the nearby galaxy NGC 1275 contains a very compact nuclear component that is opaque below about 20 GHz. The radio galaxy 3C 123 is transparent at all plotted frequencies, and its spectrum steepens above a few GHz. The quasar 3C 48 is synchrotron self-absorbed only below 100 MHz, while the quasar 3C 454.3 contains structures that become opaque at widely differing frequences. [Kellermann, K. I., & Owen, F. N. 1988, in Galactic and Extragalactic Radio Astronomy, eds. G. L. Verschuur & K. I. Kellermann (Springer Verlag)]