Chapter 4

Synchrotron emission and absorption

4.1 Introduction

We now know for sure that many astrophysical sources are magnetized and have relativistic leptons. Magnetic field and relativistic particles are the two ingredients to have synchrotron radiation. What is responsible for this kind of radiation is the Lorentz force, making the particle to gyrate around the magnetic field lines. Curiously enough, this force *does not work*, but makes the particles to accelerate even if their velocity modulus hardly changes.

The outline of this section is:

1. We will derive the total power emitted by the single electron. Total means integrated over frequency and over emission angles. This will require to generalize the Larmor formula to the relativistic case;

2. We will then outline the basics of the spectrum emitted by the single electron. This is treated in several text–books, so we will concentrate on the basic concepts;

3. Spectrum from an ensemble of electrons. Again, only the basics;

4. Synchrotron self absorption. We will try to discuss things from the point of view of a photon, that wants to calculate its survival probability, and also the point of view of the electron, that wants to calculate the probability to absorb the photon, and then increase its energy and momentum.

4.2 Total losses

To calculate the total (=integrated over frequencies and emission angles) synchrotron losses we go into the frame that is instantaneously at rest with the particle (in this frame v is zero, but not the acceleration!). This is because we will use the fact that the *emitted* power is Lorentz invariant:

$$P_{\rm e} = P'_{\rm e} = \frac{2e^2}{3c^3}a'^2 = \frac{2e^2}{3c^3}\left[a'^2_{\parallel} + a'^2_{\perp}\right] \tag{4.1}$$

where the subscript "e" stands for "emitted". The fact that the power is invariant sounds natural, since after all, power is energy over time, and both energy and time transforms the same way (in special relativity with rulers and clocks). But be aware that this *does not mean* that the emitted and *received* power are the same. They are not!

The problem is now to find how the parallel (to the velocity vector) and perpendicular components of the acceleration Lorentz transform. This is done in text books, so we report the results:

$$\begin{aligned} a'_{\parallel} &= \gamma^3 a_{\parallel} \\ a'_{\perp} &= \gamma^2 a_{\perp} \end{aligned} \tag{4.2}$$

where γ is the particle Lorentz factor. One easy way to understand and remember these transformations is to recall that the acceleration is the second derivative of space with respect to time. The perpendicular component of the displacement is invariant, so we have only to transform (twice) the time (factor γ^2). The parallel displacement instead transforms like γ, hence the γ^3 factor.

The generalization of the Larmor formula is then:

$$P_{\rm e} = P'_{\rm e} = \frac{2e^2}{3c^3}\left[a'^2_{\parallel} + a'^2_{\perp}\right] = \frac{2e^2}{3c^3}\gamma^4\left[\gamma^2 a^2_{\parallel} + a^2_{\perp}\right] \tag{4.3}$$

Don't be fooled by the γ^2 factor in front of $a^2_{\parallel}$... this component of the power is hardly important: since the velocity, for relativistic particles, is always close to c, it implies that one can get very very small acceleration in the same direction of the velocity. This is why linear accelerators minimize radiation losses. For synchrotron machines, instead, the losses due to radiation can be the limiting factor, and they are of course due to $a_{\perp}$: changing the *direction* of the velocity means large accelerations, even without any change in the velocity modulus. To go further, we have to calculate the two components of the acceleration for an electron moving in a magnetic field. Its trajectory, in general, will have an helical shape of radius $r_{\rm L}$ (the Larmor radius). The angle that the velocity vector makes with the magnetic field line is called *pitch angle*. Let us denote it with θ. We can anticipate that, in the absence of electric field and for a homogeneous magnetic field, the modulus of the velocity will not change: the magnetic field does not work, and so there is no change of energy, except for the losses due to the synchrotron radiation itself. So one important assumption is that *at least during one gyration, the*

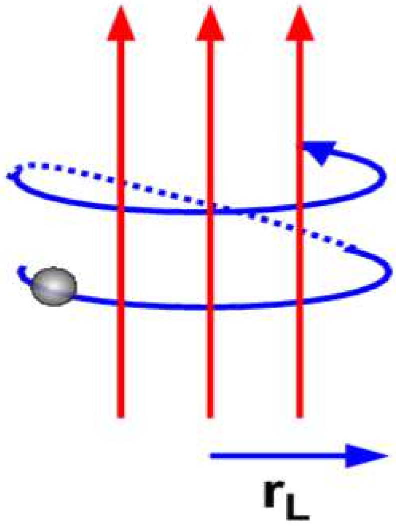

Figure 4.1: A particle gyrates along the magnetic field lines. Its trajectory has an helicoidal shape, with Larmor radius $r_{\rm L}$ and pitch angle θ.

losses are not important. This is almost always satisfied in astrophysical settings, but there are indeed some cases where this is not true.

When there is no electric field the only acting force is the (relativistic) Lorentz force:

$$F_{\rm L} = \frac{d}{dt}(\gamma m \mathbf{v}) = \frac{e}{c}\,\mathbf{v}\times\mathbf{B} \tag{4.4}$$

The parallel and perpendicular components are

$$\begin{aligned} F_{\rm L\parallel} &= e\,v_{\parallel} B = 0 \quad \rightarrow \quad a_{\parallel} = 0 \\ F_{\rm L\perp} = \gamma m \frac{dv_{\perp}}{dt} &= e\,\frac{v_{\perp}}{c} B \quad \rightarrow \quad a_{\perp} = \frac{evB\sin\theta}{\gamma mc} \end{aligned} \tag{4.5}$$

We can also derive the Larmor radius $r_{\rm L}$ by setting $a_{\perp} = v_{\perp}^2/r_{\rm L}$, and so

$$r_{\rm L} = \frac{v_{\perp}^2}{a_{\perp}} = \frac{\gamma mc^2\beta\sin\theta}{eB} \tag{4.6}$$

The fundamental frequency is the inverse of the time occurring to complete one orbit (gyration frequency), so $\nu_{\rm B} = c\beta\sin\theta/(2\pi r_{\rm L})$, giving

$$\nu_{\rm B} = \frac{eB}{2\pi\gamma mc} = \frac{\nu_{\rm L}}{\gamma} \tag{4.7}$$

where $\nu_{\rm L}$ is the Larmor frequency, namely the gyration frequency for sub–relativistic particles. Larger B means smaller $r_{\rm L}$, hence greater gyration frequencies. Vice–versa, larger γ means larger inertia, thus larger $r_{\rm L}$, and smaller gyration frequencies. Substituting $a_\perp$ given in Eq. 4.5 in the generalized Larmor formula (Eq. 4.3) we get:

$$P_{\rm S} = \frac{2e^4}{3m^2c^3} B^2\gamma^2\beta^2 \sin^2\theta \qquad (4.8)$$

We can make it nicer (for future use) by recalling that:

- The magnetic energy density is $U_B \equiv B^2/(8\pi)$
- the quantity $e^2/(m_{\rm e}c^2)$, in the case of electrons, is the classical electron radius r_0
- the square of the electron radius is proportional to the Thomson scattering cross section $\sigma_{\rm T}$, i.e. $\sigma_{\rm T} = 8\pi r_0^2/3 = 6.65 \times 10^{-25}$ cm^2.

Making these substitutions, we have that the synchrotron power emitted by a single electron of given pitch angle is:

$$P_{\rm S}(\theta) = 2\sigma_{\rm T} c U_B \gamma^2\beta^2 \sin^2\theta \qquad (4.9)$$

In the case of an *isotropic distribution of pitch angles* we can average the term $\sin^2\theta$ over the solid angle. The result is 2/3, giving

$$\langle P_{\rm S}\rangle = \frac{4}{3}\sigma_{\rm T} c U_B \gamma^2\beta^2 \qquad (4.10)$$

Now pause, and ask yourself:

- Is P_S valid only for relativistic particles, or does it describe correctly the radiative losses also for sub–relativistic ones?
- In the relativistic case the losses are proportional to the *square* of the electron energy. Do you understand why? And for sub–relativistic particles?
- What happens of we have protons, instead of electrons?
- What happens for $\theta \to 0$? Are you sure? (that losses vanishes..). Ok, but what happens to the *received* power when you have the lines of the magnetic field along the line of sight, and a beam of particles, all with a small pitch angles, shooting at you?
- Why on earth there is the scattering cross section? Is this a coincidence or does it hide a deeper fact?

4.2.1 Synchrotron cooling time

When you want to estimate a timescale of a quantity A, you can always write $t = A/\dot{A}$. In our case A is the energy of the particle. For electrons with an isotropic pitch angle distribution we have

$$t_{\rm syn} = \frac{E}{\langle P_{\rm S}\rangle} = \frac{\gamma m_{\rm e}c^2}{(4/3)\sigma_{\rm T}cU_B\gamma^2\beta^2} \sim \frac{7.75\times10^8}{B^2\gamma}\ {\rm s} = \frac{24.57}{B^2\gamma}\ {\rm yr} \quad (4.11)$$

In the vicinity of a supermassive AGN black hole we can have $B = 10^3 B_3$ Gauss and $\gamma = 10^3\gamma_3$, yielding $t_{\rm syn} = 0.75/(B_3^2\gamma_3)$ s. The same electron, in the radio lobes of a radio loud quasar with $B = 10^{-5}B_{-5}$ Gauss, cools in $t_{\rm syn} = 246$ million years.

4.3 Spectrum emitted by the single electron

4.3.1 Basics

There exists a typical frequency associated to the synchrotron process. This is related to the inverse of a typical time. If the electron is relativistic, this is *not* the revolution period. Instead, it is the fraction of the time, for each orbit, during which the observer *receives* some radiation. To simplify, consider an electron with a pitch angle of 90°, and look at Fig. 4.2, illustrating the typical patterns of the produced radiation for sub–relativistic electrons moving with a velocity parallel (top panel) or perpendicular (mid panel) to the acceleration. In the bottom panel we see the pattern for a relativistic electron (with $\mathbf{v}\perp\mathbf{a}$): it is strongly beamed in the forward direction. This is the direct consequence of the aberration of light, making half of the photons be emitted in a cone of semi–aperture angle $1/\gamma$ (which is called *the beaming angle*). Note that this *does not* mean that *half of the power* is emitted within $1/\gamma$, because the photons inside the beaming cone are more energetic than those outside, and are more tightly packed (do you remember the δ^4 factor when studying beaming?). 1

To go further, recall what we do when we study a time series and we want to find the power spectrum: we Fourier transform it. In this case we must do the same. Therefore if there is a typical timescale during which we *receive* most of the signal, we can say that most of the power is emitted at a frequency that is the inverse of that time.

Look at Fig. 4.3: the relativistic electron emits photons all along its orbit, but it will "shoot" in a particular direction only for the time

$$\Delta t_{\rm e} \sim \frac{AB}{v} = \frac{1}{v}r_{\rm L}\frac{\gamma}{2} = \frac{1}{v}\frac{2mcv}{eB} = \frac{2}{2\pi}\frac{1}{\nu_{\rm L}} = \frac{2}{2\pi}\frac{1}{\gamma\nu_{\rm B}} \quad (4.12)$$

This is the *emitting* time during which the electron emits photons that will reach the observer. We can approximate the arc AB with a straight segment

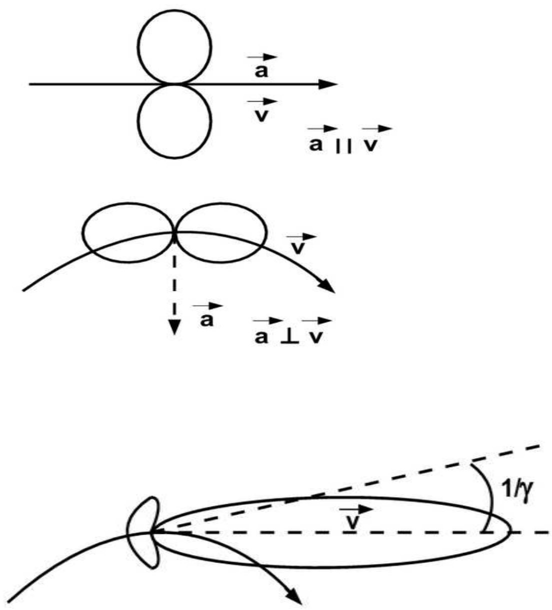

Figure 4.2: Radiation patterns for a non relativistic particle with the velocity parallel (top) or perpendicular (mid) to the acceleration. When the particle is relativistic, the pattern strongly changes due to the aberration of light, and is strongly beamed in the forward direction.

if the electron is relativistic, and the observed will then measure an *arrival* time Δt_A that is shorter than Δt_e:

$$\Delta t_{\rm A} = \Delta t_{\rm e}\,(1-\beta) = \Delta t_{\rm e}\,\frac{(1-\beta^2)}{1+\beta} \sim \frac{\Delta t_{\rm e}}{2\gamma^2} = \frac{1}{2\pi\gamma^3\nu_{\rm B}} \tag{4.13}$$

The inverse of this time is the typical synchrotron (angular) frequency $\omega_{\rm s} = 2\pi\nu_{\rm s}$, so that:

$$\nu_{\rm s} = \frac{1}{2\pi\Delta t_{\rm A}} = \gamma^3\nu_{\rm B} = \gamma^2\nu_{\rm L} = \gamma^2\,\frac{eB}{2\pi m_e c} \tag{4.14}$$

This is a factor γ^3 greater than the fundamental frequency, and a factor γ^2 greater than the Larmor frequency, defined as the typical frequency of non–relativistic particles. We expect that the particle emits most of its power at this frequency.

4.3.2 The real stuff

One can look at any text book for a detailed discussion of the procedure to calculate the spectrum emitted by the single particle. Here we report

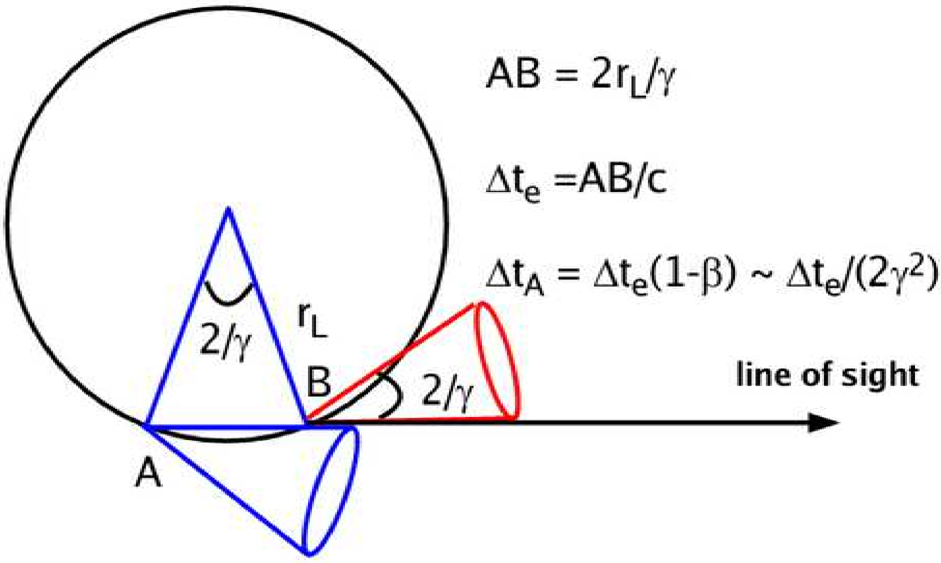

Figure 4.3: A relativistic electron is gyrating along a magnetic field line with pitch angle 90°. Its trajectory is then a circle of radius $r_{\rm L}$. Due to aberration, an observer will "see it" (i.e. will measure an electric field) when the beaming cone of total aperture angle $2/\gamma$ is pointing at him.

the results: the power per unit frequency emitted by an electron of given Lorentz factor and pitch angle is:

$$\begin{aligned} P_{\rm s}(\nu,\gamma,\theta) &= \frac{\sqrt{3}e^3 B\sin\theta}{m_{\rm e}c^2}\,F(\nu/\nu_c) \\ F(\nu/\nu_c) &\equiv \frac{\nu}{\nu_c}\int_{\nu/\nu_c}^{\infty} K_{5/3}(y)dy \\ \nu_c &\equiv \frac{3}{2}\,\nu_{\rm s}\sin\theta \end{aligned} \tag{4.15}$$

This is the power integrated over the emission pattern. $K_{5/3}(y)$ is the modified Bessel function of order 5/3. The dependence upon frequency is contained in $F(\nu/\nu_c)$, that is plotted in Fig. 4.4. This function peaks at $\nu \sim 0.29\nu_c$, therefore very close to what we have estimated before, in our very approximate treatment. The low frequency part is well approximated by a power law of slope 1/3:

$$F(\nu/\nu_c) \to \frac{4\pi}{\sqrt{3}\Gamma(1/3)}\left(\frac{\nu}{2\nu_c}\right)^{1/3} \quad (\nu \ll \nu_c) \tag{4.16}$$

At $\nu \gg \nu_c$ the function decays exponentially, and can be approximated by:

$$F(\nu/\nu_c) \to \left(\frac{\pi}{2}\right)^{1/2}\left(\frac{\nu}{\nu_c}\right)^{1/2} e^{-\nu/\nu_c} \quad (\nu \gg \nu_c) \tag{4.17}$$

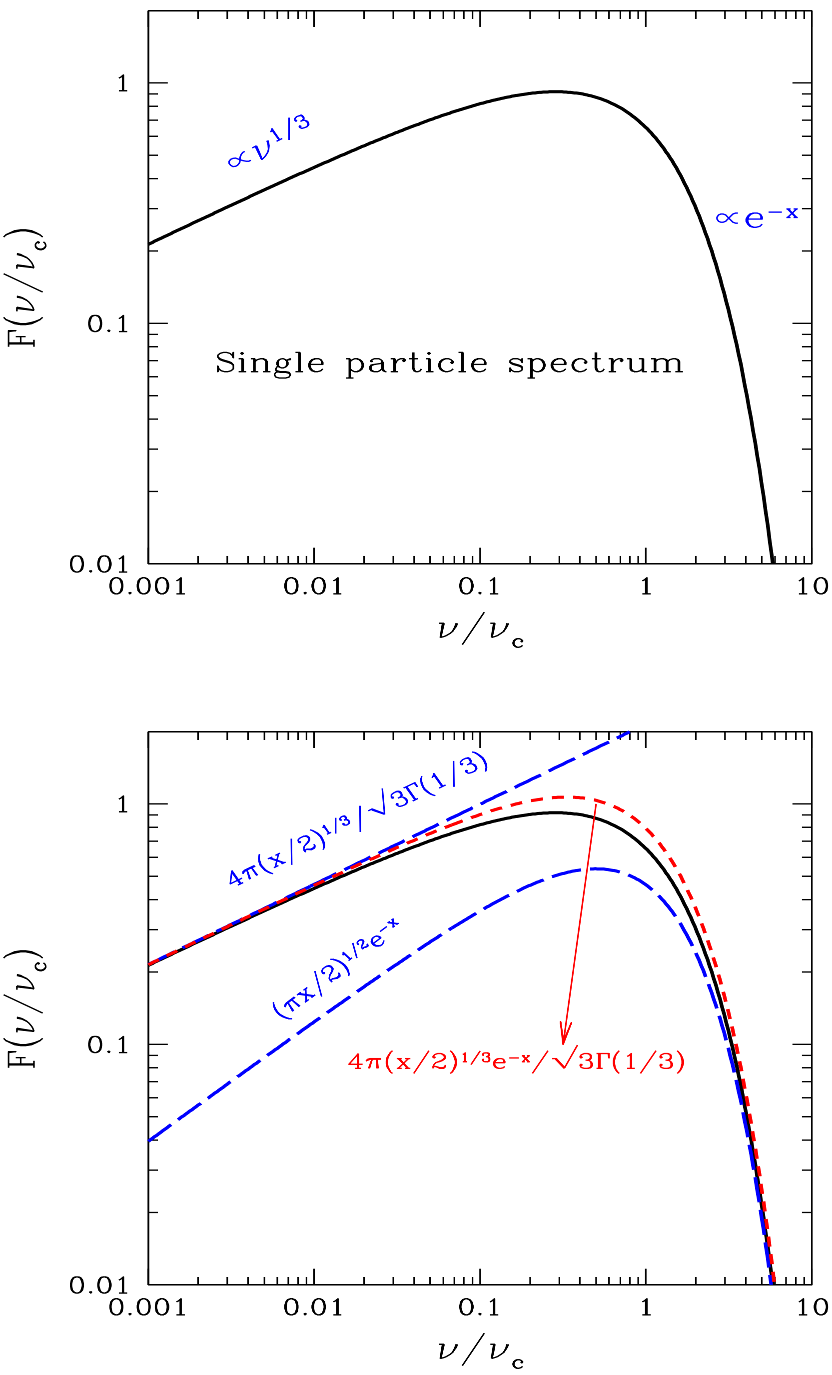

Figure 4.4: Top panel: The function $F(\nu/\nu_c)$ describing the synchrotron spectrum emitted by the single electron. Bottom panel: $F(\nu/\nu_c)$ is compared with some approximating formulae, as labeled. We have defined $x \equiv \nu/\nu_c$.

Another approximation valid for all frequency, but overestimating F around the peak, is:

$$F(\nu/\nu_c) \sim \frac{4\pi}{\sqrt{3}\Gamma(1/3)} \left(\frac{\nu}{2\nu_c}\right)^{1/3} e^{-\nu/\nu_c} \tag{4.18}$$

4.3.3 Limits of validity

One limit can be obtained by requiring that, during one orbit, the emitted energy is much smaller than the electron energy. If not, the orbit is modified, and our calculations are no more valid. For non–relativistic electrons this translates in demanding that

$$h\nu_{\rm B} < m_{\rm e}c^2 \to B < \frac{2\pi m_{\rm e}^2 c^3}{he} \equiv B_{\rm c} \tag{4.19}$$

where $B_{\rm c} \sim 4.4\times 10^{13}$ Gauss is the critical magnetic field, around and above which quantum effects appears (i.e. quantized orbits, Landau levels and so on).

For relativistic particles we demand that the energy emitted during one orbit does not exceed the energy of the particle.

$$\frac{P_{\rm s}}{\nu_{\rm B}} < \gamma m_{\rm e}c^2 \to B < \frac{e/\sigma_{\rm T}}{\gamma^2 \sin^2\theta} \sim \frac{7.22\times 10^{14}}{\gamma^2\sin^2\theta} \text{ Gauss} \tag{4.20}$$

Therefore for large γ we reach the validity limit even if the magnetic field is sub–critical.

For very small pitch angles beware that the spectrum is not described by $F(\nu/\nu_c)$, but consists of a blue-shifted cyclotron line. This is because, in the gyroframe, the particle is sub–relativistic, and so it emits only one (or very few) harmonics, that the observer sees blueshifted.

4.3.4 From cyclotron to synchrotron emission

A look at Fig. 4.5 helps to understand the difference between cyclotron and synchrotron emission. When the particle is very sub–relativistic, the observed electric field is sinusoidal in time. Correspondingly, the Fourier transform of $E(t)$ gives only one frequency, the first harmonic. Increasing somewhat the velocity (say, $\beta \sim 0.01$) the emission pattern starts to be asymmetric (for light aberration) and as a consequence $E(t)$ must be described by more than just one sinusoid, and higher order harmonics appear. In these cases the ratio of the power contained in successive harmonics goes as β^2.

Finally, for relativistic (i.e. $\gamma \gg 1$) particles, the pattern is so asymmetric that the observers sees only spikes of electric field. They repeat themselves with the gyration period, but all the power is concentrated into $\Delta t_{\rm A}$. To reproduce $E(t)$ in this case with sinusoids requires a large number

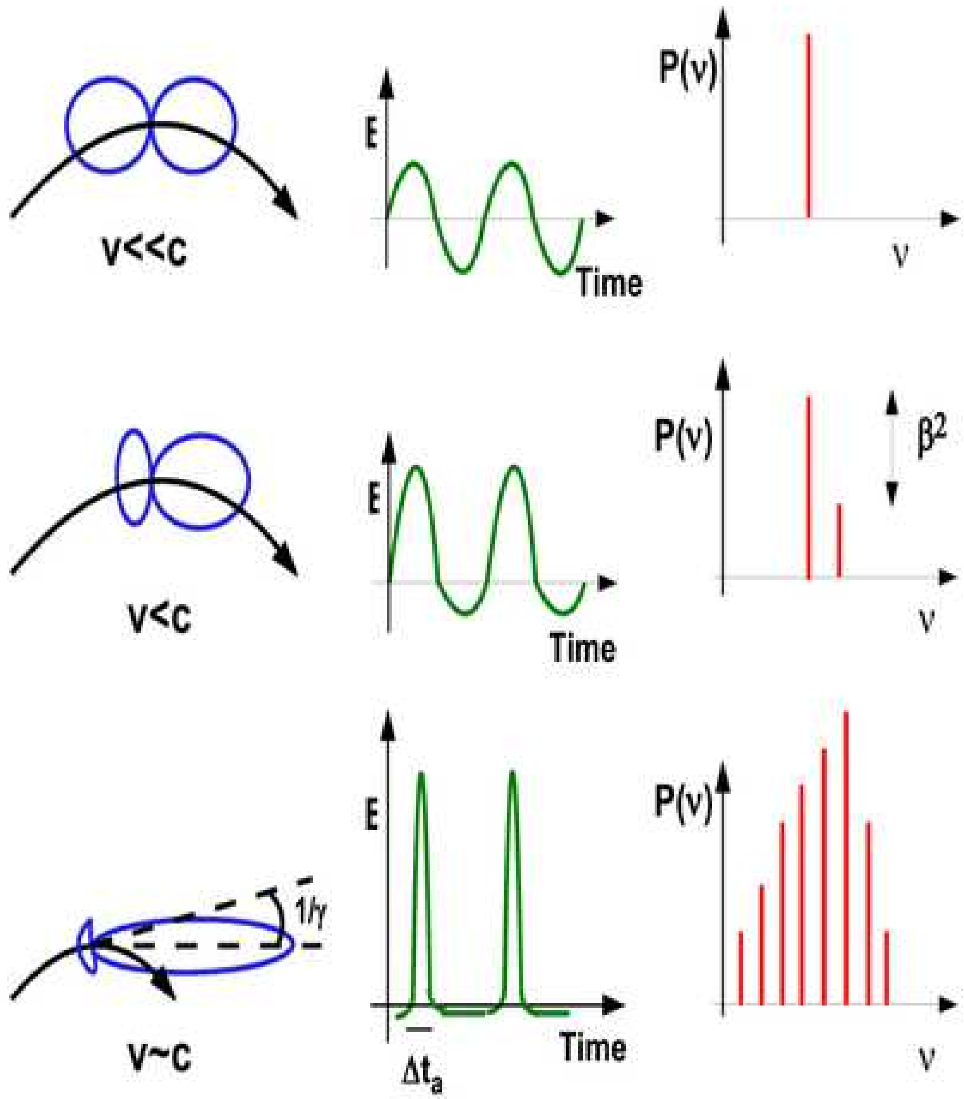

Figure 4.5: From cyclo to synchro: if the emitting particle has a very small velocity, the observer sees a sinusoidal (in time) electric field $E(t)$. Increasing the velocity the pattern becomes asymmetric, and the second harmonic appears. For $0 < \beta \ll 1$ the power in the second harmonic is a factor β^2 less than the power in the first. For relativistic particles, the pattern becomes strongly beamed, the emission is concentrated in the time Δt_A. As a consequence the Fourier transformation of $E(t)$ must contain many harmonics, and the power is concentrated in the harmonics of frequencies $\nu \sim 1/\Delta t_A$. Broadening of the harmonics due to several effects ensures that the spectrum in this case becomes continuous. Note that the fundamental harmonic becomes *smaller* increasing γ (since $\nu_B \propto 1/\gamma$).

of them, with frequencies going at least up to $1/\Delta t_A$. In this case the harmonics are many, guaranteeing that the spectrum becomes continuous with any reasonable line broadening effect, and the power is concentrated at high frequencies.

4.4 Emission from many electrons

Again, this problem is treated in several text books, so we repeat the basic results using some approximations, tricks and shortcuts.

The queen of the particle energy distributions in high energy astrophysics is the *power law* distribution:

$$N(\gamma) = K\gamma^{-p} = N(E)\frac{dE}{d\gamma}; \qquad \gamma_{\rm min}\gamma < \gamma_{\rm max} \tag{4.21}$$

Now, assuming that the distribution of pitch angles is the same at low and high γ, we want to obtain the synchrotron emissivity produced by these particles. Beware that the *emissivity* is the power per unit *solid angle* produced within 1 cm^3. The specific emissivity is also per unit of frequency. So, if Eq. 4.21 represents a density, we should integrate over γ the power produced by the single electron (of a given γ) times $N(\gamma)$, and divide all it by 4π, if the emission is isotropic:

$$\epsilon_{\rm s}(\nu,\theta) = \frac{1}{4\pi}\int_{\gamma_{\rm min}}^{\gamma_{\rm max}} N(\gamma)P(\gamma,\nu,\theta)d\gamma \tag{4.22}$$

Doing the integral one easily finds that, in an appropriate range of frequencies:

$$\epsilon_{\rm s}(\nu,\theta) \propto KB^{(p+1)/2}\nu^{-(p-1)/2} \tag{4.23}$$

The important thing is that a power law electron distribution produces a power law spectrum, and the two spectral indices are related. We traditionally call α the spectral index of the radiation, namely $\epsilon_{\rm s} \propto \nu^{-\alpha}$. We then have

$$\alpha = \frac{p-1}{2} \tag{4.24}$$

This result is so important that it is worth to try to derive it in a way as simple as possible, even without doing the integral of Eq. 4.22. We can in fact use the fact that the synchrotron spectrum emitted by the single particle is peaked. We can then say, without being badly wrong, that all the power is emitted at the typical synchrotron frequency:

$$\nu_{\rm s} = \gamma^2\nu_{\rm L}; \qquad \nu_{\rm L} \equiv \frac{eB}{2\pi m_{\rm e}c} \tag{4.25}$$

In other words, there is a tight correspondence between the energy of the electron and the frequency it emits. To simplify further, let us assume that the pitch angle is 90°. The emissivity at a given frequency, within an interval $d\nu$, is then the result of the emission of electrons having the appropriate energy γ, within the interval $d\gamma$

$$\epsilon_{\rm s}(\nu)d\nu \; = \; \frac{1}{4\pi}\, P_{\rm s} N(\gamma)d\gamma; \quad \gamma = \left(\frac{\nu}{\nu_{\rm L}}\right)^{1/2}; \quad \frac{d\gamma}{d\nu} = \frac{\nu^{-1/2}}{2\nu_{\rm L}^{1/2}} \tag{4.26}$$

we then have

$$\begin{aligned} \epsilon_{\rm s}(\nu) &\propto B^2\gamma^2 K\gamma^{-p}\frac{d\gamma}{d\nu} \\ &\propto B^2 K \left(\frac{\nu}{\nu_{\rm L}}\right)^{(2-p)/2} \frac{\nu^{-1/2}}{\nu_{\rm L}^{1/2}} \\ &\propto K\, B^{(p+1)/2}\, \nu^{-(p-1)/2} \end{aligned} \tag{4.27}$$

where we have used $\nu_{\rm L} \propto B$.

The synchrotron flux received from a homogeneous and thin source of volume $V \propto R^3$, at a distance $d_{\rm L}$, is

$$\begin{aligned} F_{\rm s}(\nu) &= 4\pi\epsilon_{\rm s}(\nu)\frac{V}{4\pi d_{\rm L}^2} \\ &\propto \frac{R^3}{d_{\rm L}^2}\, KB^{1+\alpha}\nu^{-\alpha} \\ &\propto \theta_s^2 RKB^{1+\alpha}\nu^{-\alpha} \end{aligned} \tag{4.28}$$

where θ_s is the angular radius of the source (not the pitch angle!). Observing the source at two different frequencies allows to determine α, hence the slope of the particle energy distribution. Furthermore, if we know the distance and R, the normalization depends on the particle density and the magnetic field: two unknowns and only one equation. We need another relation to close the system. As we will see in the following, this is provided by the self–absorbed flux.

4.5 Synchrotron absorption: photons

All emission processes have their absorption counterpart, and the synchrotron emission is no exception. What makes synchrotron special is really the fact that it is done by relativistic particles, and they are almost never distributed in energy as a Maxwellian. If they were, we could use the well known fact that the ratio between the emissivity and the absorption coefficient is equal to the black body (Kirchhoff law) and then we could easily find the absorption coefficient. But in the case of a non–thermal particle distribution we cannot do that. Instead we are obliged to go back to more fundamental relations, the one between the A and B Einstein coefficients relating spontaneous and stimulated emission and "true" absorption (by the way, recall that the absorption coefficient is what remains subtracting stimulated emission from "true" absorption). But we once again will use some tricks, in order to be as simple as possible. These are the steps:

1. The first trick is to think to our power law energy distribution as a superposition of Maxwellians, of different temperatures. So, we will relate the energy $\gamma m_e c^2$ of a given electron to the energy kT of a Maxwellian.

2. We have already seen that there is a tight relation between the emitted frequency and γ. Since the emission and absorption processes are related, we will assume that a particular frequency ν is preferentially absorbed by those electrons that can emit it.

3. As a consequence, we can associate our "fake" temperature to the frequency:
$$kT \sim \gamma m_e c^2 \sim m_e c^2 \left(\frac{\nu}{\nu_L}\right)^{1/2} \tag{4.29}$$

4. For an absorbed source the *brightness temperature* T_b, defined by
$$I(\nu) \equiv 2kT_b \frac{\nu^2}{c^2} \tag{4.30}$$
must be equal to the kinetic "temperature" of the electrons, and so
$$\begin{aligned} I(\nu) &\equiv 2kT\frac{\nu^2}{c^2} \sim 2m_e\,\nu^2 \left(\frac{\nu}{\nu_L}\right)^{1/2} \\ &\propto \frac{\nu^{5/2}}{B^{1/2}} \end{aligned} \tag{4.31}$$

These are the right dependencies. Note that the spectrum is $\propto \nu^{5/2}$, not ν^2, and this is the consequence of having "different temperatures". Note also that the particle density disappeared: if you think about it is natural: the more electrons you have, the more you emit, but the more you absorb. Finally, even the *slope* of the particle distribution is not important, it controls (up to a factor of order unity) only the normalization of $I(\nu)$ (our ultra–simple derivation cannot account for it, see the Appendix).

The above is valid as long as we can associate a specific γ to any ν. This is not always the case. Think for instance to a cut–off distribution, with $\gamma_{min} \gg 1$. In this case the electrons with γ_{min} are the most efficient emitters and absorbers of all photons with $\nu < \nu_{min} \equiv \gamma^2_{min}\nu_L$. So in this case we *should not* associate a different temperature when dealing with different

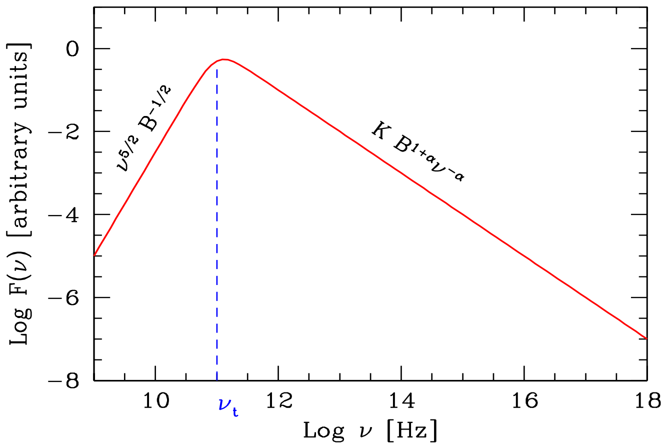

Figure 4.6: The synchrotron spectrum from a partially self absorbed source. Observations of the self absorbed part could determine B. Observations of the thin part can then determine K and the electron slope p.

$\nu < \nu_{\rm min}$. But if do not change T, we recover a self–absorbed intensity $I(\nu) \propto \nu^2$ (i.e. Raleigh–Jeans like).

Now, going from the intensity to the flux, we must integrate $I(\nu)$ over the angular dimension of the source (i.e. θ_s), obtaining

$$F(\nu) \propto \theta_s^2 \frac{\nu^{5/2}}{B^{1/2}} \tag{4.32}$$

if we could observe a self–absorbed source, of known angular size, we could then derive its magnetic field *even without knowing its distance*.

4.5.1 From thick to thin

To describe the transition from the self absorbed to the thin regime we have to write the radiation transfer equation. The easiest one is for a slab. Calling κ_ν the specific absorption coefficient [cm^{-1}] we have

$$I(\nu) = \frac{\epsilon(\nu)}{\kappa_\nu}(1 - e^{-\tau_\nu}); \quad \tau_\nu \equiv R\kappa_\nu \tag{4.33}$$

it is instructive to write Eq. 4.33 in the form:

$$I(\nu) = \epsilon(\nu) R \, \frac{1 - e^{-\tau_\nu}}{\tau_\nu} \tag{4.34}$$

because in this way it is evident that when $\tau_\nu \gg 1$ (self absorbed regime), we simply have

$$I(\nu) = \frac{\epsilon(\nu)R}{\tau_\nu} = \frac{\epsilon(\nu)}{\kappa_\nu}; \quad \tau_\nu \gg 1 \tag{4.35}$$

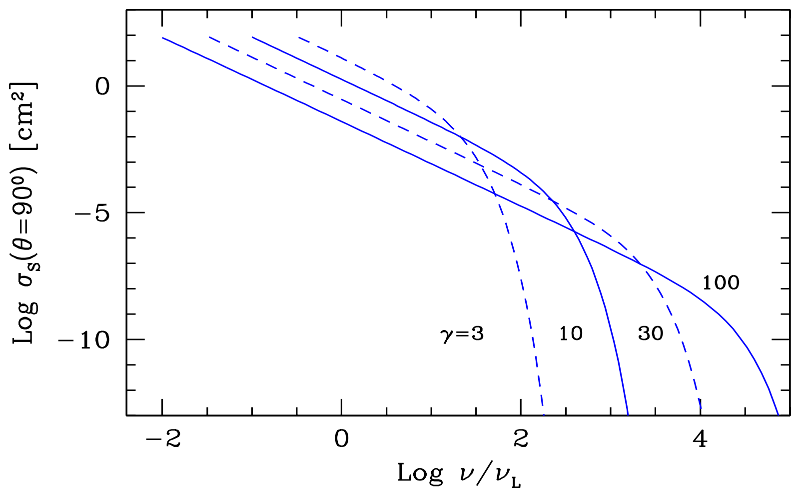

Figure 4.7: The synchrotron absorption cross section as a function of ν/ν_L for different values of γ, as labeled, assuming a pitch angle of $\theta = 90°$ and a magnetic field of 1 Gauss.

One can interpret it saying that the intensity is coming from electrons lying in a shell within R/τ_ν from the surface.

Since we have already obtained $I(\nu) \propto \nu^{5/2}B^{-1/2}$ in the absorbed regime, we can derive the dependencies of the absorption coefficient:

$$\kappa_\nu = \frac{\epsilon(\nu)}{I(\nu)} \propto \frac{KB^{(p+1)/2}\nu^{-(p-1)/1}}{\nu^{5/2}B^{-1/2}} = KB^{(p+2)/2}\nu^{-(p+4)/2} \qquad (4.36)$$

Note the rather strong dependence upon frequency: at large frequencies, absorption is small.

The obvious division between the thick and thin regime is when $\tau_\nu = 1$. We call *self–absorption frequency*, ν_t, the frequency when this occurs. We then have:

$$\tau_{\nu_t} = R\kappa_{\nu_t} = 1 \rightarrow \nu_t \propto \left[RK\, B^{(p+2)/2}\right]^{2/(p+4)} \qquad (4.37)$$

The self–absorption frequency is a crucial quantity for studying synchrotron sources: part of the reason is that it can be thought to belong to both regimes (thin and thick), the other reason is that the synchrotron spectrum peaks very close to ν_t (see Fig. 4.6) even if not exactly at ν_t (see the Appendix).

4.6 Synchrotron absorption: electrons

In the previous section we have considered what happens to the emitted spectrum when photons are emitted and absorbed. This is described by the absorption coefficient. But now imagine to be an electron, that emits and absorbs synchrotron photons. You would probably be interested if your budget is positive or negative, that is, if you are loosing or gaining energy. This is most efficiently described by a cross section, that tells you the probability to absorb a photon. Surprisingly, the synchrotron absorption cross section has been derived relatively recently (Ghisellini and Svensson 1991), and its expression is:

$$\sigma_{\rm s}(\nu,\gamma,\theta) \;=\; \frac{16\pi^2}{3\sqrt{3}}\,\frac{e}{B}\,\frac{1}{\gamma^5\sin\theta}K_{5/3}\left(\frac{\nu}{\nu_{\rm c}\sin\theta}\right) \qquad (4.38)$$

For frequencies $\nu \ll \nu_{\rm c}$ this expression can be approximated by:

$$\sigma_{\rm s}(\nu,\gamma,\theta) \;=\; \frac{8\pi^2(3\sin\theta)^{2/3}\Gamma(5/3)}{3\sqrt{3}}\,\frac{e}{B}\left(\frac{\nu}{\nu_{\rm L}/\gamma}\right)^{-5/3}; \quad \nu \ll \nu_{\rm c} \qquad (4.39)$$

Note these features:

- At the fundamental frequency $\nu_{\rm L}/\gamma$, the cross section does not depend on γ.
- The dimensions are given by e/B: this factor is proportional to the product of the classical electron radius and the Larmor wavelength (or radius). Imagine an electron with 90° pitch angle, and to see its orbit from the side: you would see a rectangle of base $r_{\rm L}$ and height r_0. The area of this rectangle is of the order of e/B. At low frequencies, $\sigma_{\rm s}$ can be orders of magnitudes larger than the Thomson scattering cross section.
- There is no explicit dependence on the particle mass. However, protons have much smaller $\nu_{\rm L}$, and the dependence on mass is hidden there. Nevertheless, electrons and protons have the same cross section (of order e/B) at their respective fundamental frequencies.

Fig. 4.7 shows $\sigma_{\rm s}$ as a function of $\nu/\nu_{\rm L}$ for different γ. The thing it should be noticed is that this cross section is really large. Can we make some useful use of it? Well, there are at least two issues, one concerning energy, and the other concerning momentum.

First, electrons emitting and absorbing synchrotron photons do so with a large efficiency. *They can talk each other by exchanging photons.* Therefor, even if they are distributed as a power law in energy at the beginning, they will try to form a Maxwellian. They will form it, as long as other competing processes are not important, such as inverse Compton scatterings. The formation of the Maxwellian will interest only the low energy part of the electron distribution, where absorption is important. Note that this thermalization process works exactly when Coulomb collisions fail: they are inefficient at low density and high temperature, while synchrotron absorption can work for relativistic electrons even if they are not very dense.

The second issue concerns exchange of momentum between photons and electrons. Suppose that a magnetized region with relativistic electrons is illuminated by low frequency radiation by another source, located aside. The electrons will efficiently absorb this radiation, and thus its momentum. The magnetized region will then accelerate.

References

Ghisellini G. & Svensson R., 1991, MNRAS, 252, 313
Ghisellini G., Haardt F. & Svensson R., 1998, MNRAS, 297, 348

4.7 Appendix: Useful Formulae

In this section we collect several useful formulae concerning the synchrotron emission. When possible, we give also simplified analytical expressions. We will often consider that the emitting electrons have a distribution in energy which is a power law between some limits γ_1 and γ_2. Electrons are assumed to be isotropically distributed in the comoving frame of the emitting source. Their density is

$$N(\gamma) = K\gamma^{-p}; \quad \gamma_1 < \gamma < \gamma_2 \tag{4.40}$$

The Larmor frequency is defined as:

$$\nu_L \equiv \frac{eB}{2\pi m_{\rm e} c} \tag{4.41}$$

4.7.1 Emissivity

The synchrotron emissivity $\epsilon_{\rm s}(\nu, \theta)$ [erg cm^{-3} s^{-1} sterad^{-1}] is

$$\epsilon_{\rm s}(\nu, \theta) \equiv \frac{1}{4\pi} \int_{\gamma_1}^{\gamma_2} N(\gamma) P_{\rm s}(\nu, \gamma, \theta) d\gamma \tag{4.42}$$

where $P_{\rm s}(\nu, \gamma, \theta)$ is the power emitted at the frequency ν (integrated over all directions) by the single electron of energy $\gamma m_{\rm e} c^2$ and pitch angle θ. For electrons making the same pitch angle θ with the magnetic field, the emissivity is

$$\epsilon_{\rm s}(\nu,\theta) \;=\; \frac{3\sigma_{\rm T} c K U_B}{8\pi^2 \nu_L} \left(\frac{\nu}{\nu_L}\right)^{-\frac{p-1}{2}} (\sin\theta)^{\frac{p+1}{2}}\, 3^{\frac{p}{2}}\, \frac{\Gamma\left(\frac{3p-1}{12}\right)\Gamma\left(\frac{3p+19}{12}\right)}{p+1} \qquad (4.43)$$

between $\nu_1 \gg \gamma_1^2 \nu_L$ and $\nu_2 \ll \gamma_2^2 \nu_L$. If the distribution of pitch angles is isotropic, we must average the $(\sin\theta)^{\frac{p+1}{2}}$ term, obtaining

$$< (\sin\theta)^{\frac{p+1}{2}} >= \int_0^{\frac{\pi}{2}} (\sin\theta)^{\frac{p+1}{2}} \sin\theta d\theta \;=\; \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{p+5}{4}\right)}{\Gamma\left(\frac{p+7}{4}\right)} \qquad (4.44)$$

Therefore the pitch angle averaged synchrotron emissivity is

$$\epsilon_{\rm s}(\nu) \;=\; \frac{3\sigma_{\rm T} c K U_B}{16\pi\sqrt{\pi}\nu_L} \left(\frac{\nu}{\nu_L}\right)^{-\frac{p-1}{2}} f_\epsilon(p) \qquad (4.45)$$

The function $f_\epsilon(p)$ includes all the products of the Γ–functions:

$$\begin{aligned} f_\epsilon(p) \;&=\; \frac{3^{\frac{p}{2}}}{p+1} \frac{\Gamma\left(\frac{3p-1}{12}\right)\Gamma\left(\frac{3p+19}{12}\right)\Gamma\left(\frac{p+5}{4}\right)}{\Gamma\left(\frac{p+7}{4}\right)} \\ &\sim\; 3^{\frac{p}{2}} \left(\frac{2.25}{p^{2.2}} + 0.105\right) \end{aligned} \qquad (4.46)$$

where the simplified fitting function is accurate at the per cent level.

4.7.2 Absorption coefficient

The absorption coefficient $\kappa_\nu(\theta)$ [cm^{-1}] is defined as:

$$\kappa_\nu(\theta) \;\equiv\; \frac{1}{8\pi m_{\rm e} \nu^2} \int_{\gamma_1}^{\gamma_2} \frac{N(\gamma)}{\gamma^2} \frac{d}{d\gamma} \left[\gamma^2 P(\nu,\theta)\right] d\gamma \qquad (4.47)$$

Written in this way, the above formula is valid even when the electron distribution is truncated. For our power law electron distribution $\kappa_\nu(\theta)$ becomes:

$$\kappa_\nu(\theta) \;\equiv\; \frac{1}{8\pi m_{\rm e} \nu^2} \int_{\gamma_1}^{\gamma_2} \frac{N(\gamma)}{\gamma^2} \frac{d}{d\gamma} \left[\gamma^2 P(\nu,\theta)\right] d\gamma \qquad (4.48)$$

Above $\nu = \gamma_1^2 \nu_L$, we have:

$$\kappa_\nu(\theta) \;=\; \frac{e^2 K}{4 m_{\rm e} c^2} (\nu_L \sin\theta)^{\frac{p+2}{2}} \nu^{-\frac{p+4}{2}}\, 3^{\frac{p+1}{2}}\, \Gamma\left(\frac{3p+22}{12}\right) \Gamma\left(\frac{3p+2}{12}\right) \qquad (4.49)$$

Averaging over the pitch angles we have:

$$< (\sin\theta)^{\frac{p+2}{2}} >= \int_0^{\frac{\pi}{2}} (\sin\theta)^{\frac{p+2}{2}} \sin\theta d\theta \;=\; \frac{\sqrt{\pi}}{2} \frac{\Gamma\left(\frac{p+6}{4}\right)}{\Gamma\left(\frac{p+8}{4}\right)} \qquad (4.50)$$

resulting in a pitch angle averaged absorption coefficient:

$$\kappa_\nu = \frac{\sqrt{\pi} e^2 K}{8 m_e c} \nu_L^{\frac{p+2}{2}} \nu^{-\frac{p+4}{2}} f_\kappa(p) \tag{4.51}$$

where the function $f_\kappa(p)$ is:

$$\begin{aligned} f_\kappa(p) &= 3^{\frac{p+1}{2}} \frac{\Gamma\left(\frac{3p+22}{12}\right)\Gamma\left(\frac{3p+2}{12}\right)\Gamma\left(\frac{p+6}{4}\right)}{\Gamma\left(\frac{p+8}{4}\right)} \\ &\sim 3^{\frac{p+1}{2}} \left(\frac{1.8}{p^{0.7}} + \frac{p^2}{40}\right) \end{aligned} \tag{4.52}$$

The simple fitting function is accurate at the per cent level.

4.7.3 Specific intensity

Simple radiative tranfer allows to calculate the specific intensity:

$$I(\nu) = \frac{\epsilon_s(\nu)}{\kappa_\nu} \left(1 - e^{-\tau_\nu}\right) \tag{4.53}$$

where the absorption optical depth $\tau_\nu \equiv \kappa_\nu R$ and R is the size of the emitting region. When $\tau_\nu \gg 1$, the esponential term vanishes, and the intensity is simply the ratio between the emissivity and the absorption coefficient. This is the self–absorbed, ot thick, regime. In this case, since both $\epsilon_s(\nu)$ and κ_ν depends linearly upon K, the resulting self–absorbed intensity does not depend on the normalization of the particle density K:

$$I(\nu) = \frac{2 m_e}{\sqrt{3}\, \nu_L^{1/2}} f_I(p) \left(1 - e^{-\tau_\nu}\right) \tag{4.54}$$

we can thus see that the slope of the self–absorbed intensity does not depend on p. Its normalization, however, does (albeit weakly) depend on p through the function $f_I(p)$, which in the case of averaging over an isotropic pitch angle distribution is given by:

$$\begin{aligned} f_I(p) = \frac{1}{p+1} &= \frac{\Gamma\left(\frac{3p-1}{12}\right)\Gamma\left(\frac{3p+19}{12}\right)\Gamma\left(\frac{p+5}{4}\right)\Gamma\left(\frac{p+8}{4}\right)}{\Gamma\left(\frac{3p+22}{12}\right)\Gamma\left(\frac{3p+2}{12}\right)\Gamma\left(\frac{p+7}{4}\right)\Gamma\left(\frac{p+6}{4}\right)} \\ &\sim \frac{5}{4\, p^{4/3}} \end{aligned} \tag{4.55}$$

where again the simple fitting function is accurate at the level of 1 per cent.

4.7.4 Self–absorption frequency

The self–absorption frequency $\nu_{\rm t}$ is defined by $\tau_{\nu_{\rm t}} = 1$:

$$\nu_{\rm t} \,=\, \nu_L \left[\frac{\sqrt{\pi} e^2 RK}{8 m_{\rm e} c \nu_L}\, f_\kappa(p)\right]^{\frac{4}{p+4}} \,=\, \nu_L \left[\frac{\pi\sqrt{\pi}}{4}\, \frac{eRK}{B}\, f_\kappa(p)\right]^{\frac{2}{p+4}} \qquad (4.56)$$

Note that the term in parenthesis is adimensional, and since RK has units of the inverse of a surface, then e/B has the dimension of a surface. In fact we have already discussed that this is the synchrotron absorption cross section of a relativistic electron of energy $\gamma m_{\rm e} c^2$ absorbing photons at the fundamental frequency ν_L/γ.

The random Lorentz factor $\gamma_{\rm t}$ of the electrons absorbing (and emitting) photons with frequency $\nu_{\rm t}$ is $\gamma_{\rm t} \sim [3\nu_{\rm t}/(4\nu_L)]^{1/2}$.

4.7.5 Synchrotron peak

In a $F(\nu)$ plot, the synchrotron spectrum peaks close to $\nu_{\rm t}$, at a frequency $\nu_{s,p}$ given by solving

$$\frac{dI(\nu)}{d\nu} \,=\, 0 \,\rightarrow\, \frac{d}{d\nu}\left[\nu^{5/2}\left(1 - e^{-\tau_\nu}\right)\right] \,=\, 0 \qquad (4.57)$$

which is equivalent to solve the equation:

$$\exp\left(\tau_{\nu_{s,p}}\right) - \frac{p+4}{5}\tau_{\nu_{s,p}} - 1 \,=\, 0 \qquad (4.58)$$

whose solution can be approximated by

$$\tau_{\nu_{s,p}} \,\sim\, \frac{2}{5} p^{1/3} \ln p \qquad (4.59)$$